Twenty-eighth Annual Columbus State Invitational Mathematics Tournament

Sponsored by
The Columbus State University
Department of Mathematics
March 2, 2002

****

The Columbus State University Mathematics faculty welcome you to this year's tournament and to our campus. We wish you success on this test and in your future studies.

Instructions

This is a 90-minute, 50-problem, multiple choice examination. There are five possible responses to each question. You should select the one "*best*" answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with a No. 2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response. The examination will be scored on the basis of +12 for each correct answer, −3 for each incorrect selection, and 0 for each omitted item. Each student will be given an initial score of +200.

Pre-selected problems will be used as tiebreakers for individual awards. These problems, designated with an asterisk (*), in order of consideration are: 26, 29, 35, 37, 39, 44 and 45.

Throughout the exam, $\overline{AB}$ will denote the line segment from point A to point B and AB will denote the length of $\overline{AB}$. Pre-drawn geometric figures are not necessarily drawn to scale.

Review and check your score sheet carefully. **Your student identification number and your school number must be encoded correctly on your score sheet**.

When you complete your test, bring your pencil, scratch paper and answer sheet to the test monitor. Leave the room after you have handed in your answer sheet. Please leave quietly so as not to disturb the other contestants. Do not congregate outside the doors by the testing area. You may keep your copy of the test. Your sponsor will have a copy of solutions to the test problems.

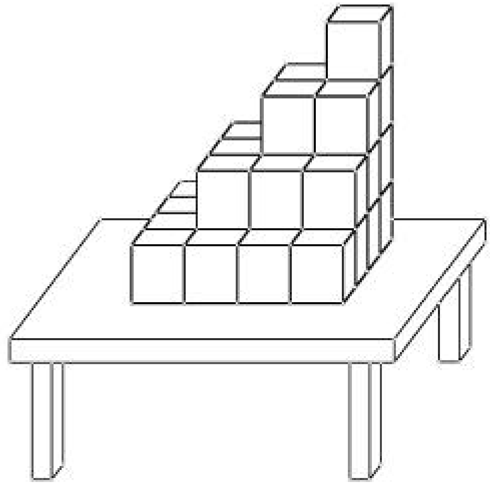
Do not open your test until instructed to do so!

1. Johnny has stacked sugar cubes on a little table as in the accompanying figure. How many sugar cubes did he use?

   (a) 50 (b) 30

   (c) 40 (d) 39

   (e) 60



2. Which of the following is an irrational number?

   (a) $\sqrt{4}$ (b) $\sqrt{2}$ (c) $\sqrt[3]{27}$

   (d) 2.71 (e) 3.141592654

3. If A is 36% of B and C is 40% of B, what is the ratio $\frac{A}{C}$?

   (a) 0.8 (b) 0.4 (c) 0.5 (d) 0.7 (e) 0.9

4. The prime factorization of the number 150 requires how many distinct prime numbers?

   (a) 4 (b) 2 (c) 0 (d) 5 (e) 3

5. Find the ratio of the area of a disk of radius 10 inches to its circumference.

   (a) 10 inches (b) $\frac{10}{\pi}$ inches (c) π inches

   (d) 5π inches (e) 5 inches

6. Let n be a positive integer. Identify the largest number from the list below.

(a) $\frac{2n}{2n+11}$ (b) $\frac{n}{n+6}$ (c) $\frac{3n}{3n+17}$ (d) $\frac{4n}{4n+23}$ (e) $\frac{5n}{5n+29}$

7. Assuming that x is not 0, 1, y or $-y$, simplify the expression $\frac{x+y}{x^2-x} \div \frac{x^2-y^2}{x^3-x^2}$.

(a) $\frac{-x}{y}$ (b) $\frac{x}{x-y}$ (c) $\frac{x}{x+y}$

(d) $\frac{1}{x(x+y)}$ (e) $\frac{x}{y-x}$

8. What is the sum of all the prime divisors of 2002?

(a) 31 (b) 102 (c) 104 (d) 152 (e) 33

9. The sides of a triangle are $\frac{4}{5}$, $\frac{1}{3}$ and $\frac{13}{15}$. Which of the following best describes this triangle?

(a) isosceles (b) acute (c) right triangle

(d) obtuse (e) equilateral

10. If $x - y + 3 = x + y - 1 = 0$ then find $5x + 3y$.

(a) 2 (b) 3 (c) 7 (d) 10 (e) 1

11. If $m > 0$ and the points $(3, m)$ and $(m, 15)$ lie on a line with slope m, what is the value of m?

(a) 3 (b) 5 (c) 2 (d) 4 (e) 1

12. The Fibonacci numbers are a sequence of numbers formed as follows: $f_1 = 1$, $f_2 = 1$, $f_3 = f_2 + f_1 = 1 + 1 = 2$, $f_4 = f_3 + f_2 = 3$, $f_5 = 3 + 2 = 5$,..., and in general $f_n = f_{n-1} + f_{n-2}$, for $n \geq 3$. Determine the value of f_{10}.

(a) 89 (b) 55 (c) 34 (d) 50 (e) 21

13. *Pascal's Triangle* is a triangular array of numbers constructed using the following pattern (notice that every number in the triangle below, which is not a 1, can be obtained by calculating the sum of the adjacent numbers on the line above it):

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

The pattern above depicts the first five complete rows of Pascal's Triangle. How many odd entries appear in the ninth row?

(a) 2 (b) 4 (c) 6 (d) 9 (e) 5

14. A sum of \$1750 is to be divided between two people in the ratio of 3 to 4. How much more money does one get than the other?

(a) \$150 (b) \$437.5 (c) \$50 (d) \$1750 (e) \$250

15. The following quadratic equations $2x^2 - 5x + 2 = 0$ and $2x^2 - 7x + 3 = 0$ have a common solution. What is the sum of the other solutions?

(a) 5 (b) 3 (c) 0 (d) 1 (e) 2

16. Solve the equation $\log_2 (\log_2 x) = 2$.

(a) 16 (b) 4 (c) 8 (d) 32 (e) 64

17. The angle $\angle A$ of a triangle $\triangle ABC$ has a measure of 60°(degrees) and the measures of the angles $\angle B$ and $\angle C$ are in a ratio of 2 to 3(respectively). Find the measure of the angle $\angle AIB$ where I is the intersection of the angle bisectors of the triangle $\triangle ABC$.

(a) 120° (b) 144° (c) 110° (d) 112° (e) 126°

18. The average weight of the 120 people in a airplane is 91 lb. There are 35 girls, 44 boys and 41 adults. If the average of the boys is 70 lb, and the average of the girls is 60 lb, what is the average of the adults?

(a) 143 (b) 140 (c) 130 (d) 123 (e) 100

19. In the Cartesian system of coordinates the point $P(x, y)$ (with coordinates x and y) has the following property: the distance from P to $A(8, 0)$ is twice the distance from P to $B(2, 0)$. What is the equation that x and y should satisfy?

(a) $x^2+2y^2-2x+102=0$ (b) $x^2+y^2-16=0$ (c) $x^2+y^2-144=0$

(d) $x^2+y^2-24x+112=0$ (e) $2x^2+y^2-2x+102=0$

20. What is the remainder when 2001^{2002} is divided by 7?

(a) 2 (b) 1 (c) 3 (d) 6 (e) 4

21. Martians have 2 legs and 3 arms. Jovians have 3 legs and 5 arms. 51 legs and 81 arms were counted at a meeting. How many more Martians than Jovians were at the meeting?

(a) 3 (b) 1 (c) 4 (d) 5 (e) 7

22. The sum of 91 consecutive integers is 2002. What is the greatest of these integers?

(a) 78 (b) 55 (c) 102 (d) 66 (e) 67

23. The sequence of digits 12345678910111213...997998999 is found by writing the counting numbers 1, 2, 3, ..., 999 in ascending order. What would be the 2002^{nd} digit from the left?

(a) 4 (b) 5 (c) 0 (d) 3 (e) 7

24. If x is not -1, 0 or 1, we define $g(x) = \frac{x-1}{x+1}$. Calculate $g(g(g(g(x))))$.

(a) $\frac{1}{x}$ (b) $-\frac{1}{x}$ (c) $2x$ (d) $-x$ (e) x

25. Which of the following has the same solution set as $|x+5| = |x-4|$?

(a) $\frac{x+5}{x-4} = 0$ (b) $x+5 = x-4$ (c) $(x+5)(x-4) = 0$

(d) $\sqrt{x+5} = \sqrt{x-4}$ (e) $(x+5)^2 = (x-4)^2$

26. $*$ What is the area bounded by the graph of the equation $26|x+3| + 77|y-4| = 2002$?

(a) 4004 (b) 2002 (c) 4000 (d) 5010 (e) 5004

27. If $f\left(\frac{2x+1}{3x+2}\right) = x$ for all real values of x except $-\frac{2}{3}$, determine $f(x)$.

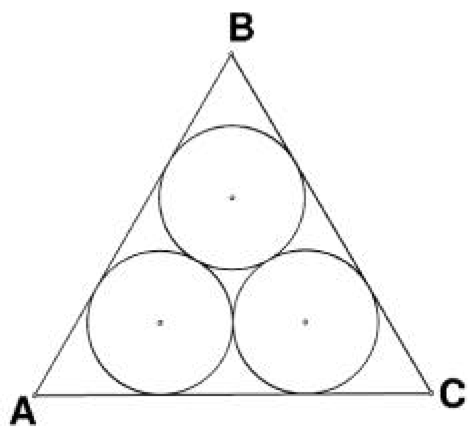
(a) $\frac{3x+2}{2x+1}$ (b) $\frac{2x-1}{2-3x}$ (c) $\frac{3x-2}{2x-1}$

(d) $\frac{3x-2}{2x+1}$ (e) $\frac{2x-1}{3x-2}$

28. Find the remainder obtained by dividing of the polynomial $x^3 - 2x^2 + 3x - 4$ by the binomial $x+1$.

(a) x (b) 2 (c) -10 (d) 3 (e) -5

29. $*$ In the accompanying figure the triangle $\triangle ABC$ has each of its sides of length 4 inches. The three circles drawn inside the triangle have the same radius and they are mutually tangent to each other. Each circle is tangent to two of the sides of the triangle. What is the radius of one of these circles?



(a) $\sqrt{3}-1$ inches (b) 0.7 inches

(c) $\frac{\sqrt{3}}{2}$ inches (d) $\frac{1}{2}$ inches

(e) 0.6 inches

30. Determine the sum of the solutions for the equation $\sqrt{11x-2}+1=\sqrt{12x+9}$.

(a) 73 (b) 51 (c) 24 (d) 30 (e) 10

31. Two fair six-sided dice are thrown. What is the probability that the product of the two numbers is a multiple of 3?

(a) $\frac{5}{9}$ (b) $\frac{1}{12}$ (c) $\frac{1}{2}$ (d) $\frac{4}{5}$ (e) $\frac{4}{9}$

32. The sum of the fourth powers of three positive integers is 2002. Find the sum of these three numbers.

(a) 20 (b) 13 (c) 10 (d) 14 (e) 12

33. A 98% concentrate is to be mixed with a mixture having a concentration of 50% to obtain 12 gallons of a mixture with concentration of 70%. How many gallons of the 98% concentrate will be needed?

(a) 4 (b) 6 (c) 7 (d) 5 (e) 8

34. What is the range of the function $f(x) = \dfrac{6x+1}{3-2x}$ defined on its maximum domain?

(a) all real numbers except -3

(b) all real numbers

(c) all real numbers except 2

(d) all real numbers except 3

(e) all real numbers except 6

35. $*$ On the diagonal $\overline{BD}$ of the rectangle $\square ABCD$, we take the point E such that $\overline{AE}$ is perpendicular to $\overline{BD}$. If $DE = 3\ in$ and $EB = 12\ in$, what is the area of the rectangle $\square ABCD$ in square inches?

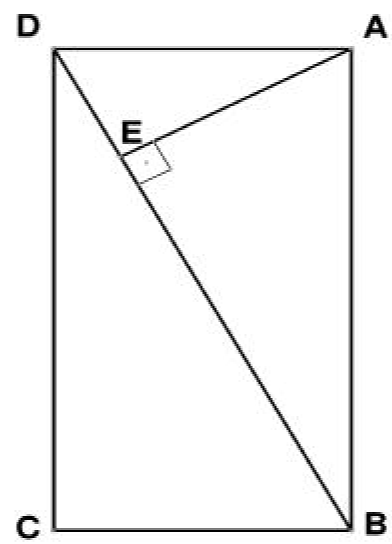
(a) 100 in^2

(b) 90 in^2

(c) 70 in^2

(d) 80 in^2

(e) 85 in^2



36. Let α be an angle in the third quadrant such that $\cos(\alpha) = -\dfrac{5}{13}$. Approximate $\tan(2\alpha)$ to three decimal places.

(a) 1.125 (b) -1.008 (c) -1.125 (d) -0.162 (e) 1.008

37. $*$ Two real numbers x and y satisfy the inequality $(x+1)^2 + (y-1)^2 \leq 2$. Which one of the following numbers is a possible value for $x + y$?

(a) 2.1

(b) 3

(c) 4

(d) -2.5

(e) none of these

38. Determine the y-intercept of the line tangent to the circle of equation $x^2 + y^2 = 1$ at the point of coordinates $\left(-\frac{3}{5}, \frac{4}{5}\right)$.

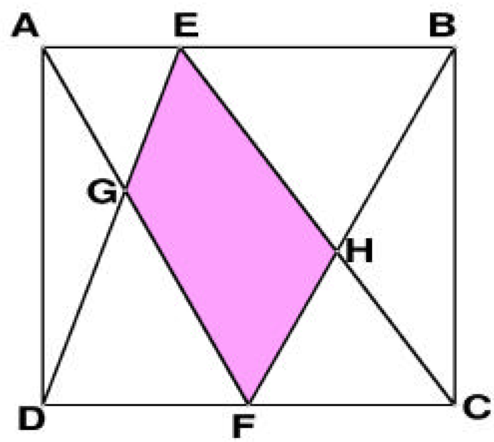
(a) 3.5 (b) 2 (c) 1.25 (d) 2.75 (e) 3

39. $*$ In the rectangle $\square ABCD$ we take the point E on $\overline{AB}$ such that $EB = 2AE$ and let F be the midpoint of $\overline{DC}$. Denote by G the intersection of $\overline{AF}$ with $\overline{DE}$ and by H the intersection of $\overline{BF}$ and $\overline{CE}$. If the area of the rectangle $\square ABCD$ is 70 square meters what is the area of the quadrilateral $\square EHFG$?

(a) 20 m^2 (b) 19 m^2

(c) 15 m^2 (d) 25 m^2

(e) 17 m^2



40. Let $f(x) = \dfrac{2x+2}{3x-1}$ defined for all real valued of x except $\dfrac{1}{3}$. Find how many points with integer coordinates are on the graph of f.

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

41. If $x + y = 1$ and $x^2 + y^2 = 3$ what is the value of $x^3 + y^3$?

(a) 3 (b) 5 (c) 1 (d) 2 (e) 4

42. Find the area of the triangle whose vertices are $(0, 0)$, $(4, 2)$ and $(1, 4)$.

(a) 4 (b) 7 (c) 5 (d) 9 (e) 6

43. Let x_1 and x_2 be the real solutions to $x^2 + ax + 1 = 0$ for some real number a. If $x_1 + 1$ and $x_2 + 1$ are solutions of the equation $x^2 - a^2x + a^2 = 0$ then calculate the sum of x_1 and x_2.

(a) -1 (b) 3 (c) -2 (d) 0 (e) 2

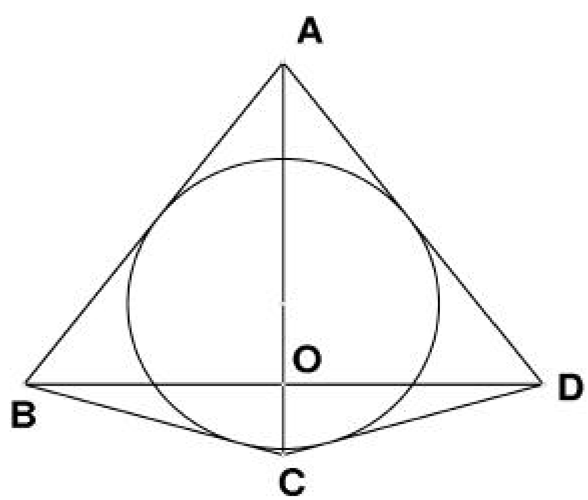
44. ∗ The function f is defined for all real values of x and it satisfies the equality

$$f(x) + xf(1-x) = 3 + x,$$

for every real number x. Find the value of $f\left(\frac{5}{3}\right)$.

(a) $\frac{1}{7}$ (b) $\frac{3}{5}$ (c) $\frac{7}{19}$ (d) $\frac{49}{19}$ (e) 0

45. ∗ The diagonals $\overline{AC}$ and $\overline{BD}$ of the quadrilateral $ABCD$ are perpendicular and they intersect at the point O. Knowing that $OB = OD = 24$, $OC = 7$ and $OA = 32$ find the radius of the circle inscribed in the quadrilateral $\square ABCD$.



(a) 13.5 (b) 14.4

(c) 16.2 (d) 14.7

(e) 15.3

46. If $\log_A B$ means the logarithm of B in base A, solve the equation $\log_9 x + \log_{x^2} 3 = 1$ for x.

(a) 9 (b) 27 (c) 81 (d) 3 (e) 1

47. If $20 \sin\alpha = 21 \cos\alpha$ and α is an angle in the first quadrant then what is the value of $\cos\alpha$?

(a) $\frac{4}{7}$ (b) $\frac{3}{5}$ (c) $\frac{20}{29}$ (d) $\frac{4}{5}$ (e) $\frac{21}{31}$

48. Let j denote a complex number that satisfies $j^2 = -1$. For which of the following values of n does $(1+j)^{2n} = 2^n j$?

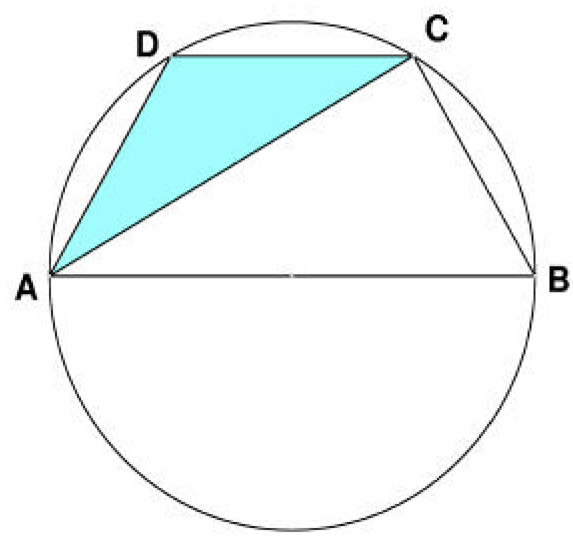
(a) 2004 (b) 2002 (c) 2001

(d) 2003 (e) none of these

49. How many ordered pairs (m, n) of positive integers are solutions of the equation $\frac{5}{m} + \frac{4}{n} = 1$?

(a) 10 (b) 3 (c) 5 (d) 4 (e) 6

50. The diameter $\overline{AB}$ of a circle (see the accompanying figure) has length of 12 units and the points C and D divide the upper half circle with endpoints at A and B into three equal arcs. What is the area of the triangle ADC (in square units)?



(a) $6\sqrt{3}$ (b) 36

(c) 16 (d) $9\sqrt{3}$

(e) 11