Solution to the First Annual Columbus State Calculus Tournament

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1. *(I) Every function f is continuous.*

   (II) Every continuous function is differentiable.

   (III) Every differentiable function is continuous.

   (IV) There exist continuous functions which are not differentiable.

   Which of the above statements are true?

   (A) (I) and (II) (B) (I) and (III) (C) (II) and (III)

   (D) (III) and (IV) (E) (I) and (IV)

Solution: (I) is false since, for instance, the function $sign(x) = \begin{cases} 1 \text{ if } x \geq 0 \\ -1 \text{ if } x < 0 \end{cases}$ is not continuous at 0. (II) is false since $f(x) = |x|$ defined for all real x is continuous, but it is not differentiable. This example also shows that (IV) is true. (III) is also true since

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

implies

$$\lim_{x \to a} f(x) = f(a).$$

Hence, the answer is D. ■

2. *The Intermediate Value Theorem (IVT) can be used for the function $f(x) = 2\sin x - \cos x$ on the interval $[a, b]$ to obtain a value $c \in (a, b)$ such that $f(c) = 0$. What is a good option for $[a, b]$?*

   (A) $[\frac{\pi}{2}, \pi]$ (B) $[\frac{\pi}{6}, \frac{\pi}{4}]$ (C) $[-\frac{\pi}{2}, 0]$ (D) $[-\frac{\pi}{4}, 0]$ (E) $[0, \pi]$

Solution: In order to apply the IVT, we need to have (in this case) $f(a)f(b) < 0$. For A, we get $f(\pi)f(\frac{\pi}{2}) = 2 > 0$. For B, we observe $f(\frac{\pi}{6})f(\frac{\pi}{4}) = (1 - \frac{\sqrt{3}}{2})\sqrt{2}/2 > 0$. Also,

$f(-\frac{\pi}{2})f(0) = 2$ and $f(-\frac{\pi}{4})f(0) = \frac{3\sqrt{2}}{2} > 0$. Finally, we have $f(0)f(\pi) = -1$, which shows that E is the right answer. ■

3. *The values of a and b are chosen in such a way the function*

$$f(x) = \begin{cases} ax^2 + bx - 1 \text{ if } x \leq -2 \\ b - ax \text{ if } x \in (-2, 2) \\ bx^2 + ax + 7 \text{ if } x \geq 2 \end{cases}$$

is continuous. What is $(a+b)^2$?

(A) 4 (B) 3 (C) 2 (D) 1 (E) 0

Solution: The function needs to be continuous at -2 which implies $f(-2) = \lim_{x \searrow -2} f(x)$. This is equivalent to $4a - 2b - 1 = b + 2a$ or $2a = 3b + 1$. Similarly, we need to have $f(2) = \lim_{x \nearrow 2} f(x)$ or $4b + 2a + 7 = b - 2a$. So, if we solve for $3b$, we obtain $3b = -4a - 7$ and then $2a = -4a - 7 + 1$. This gives $6a = -6$ or $a = -1$. Also, $3b = -4(-1) - 7 = -3$ implies $b = -1$. Then $(a+b)^2 = 4$ and therefore the answer is A. ■

4. *The function $f(x) = x^2 e^x$ is convex downward on the interval $[a, b]$, convex upward on $(-\infty, a]$ and also on $[b, \infty)$. Then what is ab ?*

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: First, we calculate the derivative $f'(x) = 2xe^x + x^2 e^x = (x^2 + 2x)e^x$ and then the second derivative $f''(x) = (2x+2)e^x + (x^2+2x)e^x$. Then $f''(x) = (x^2+4x+2)e^x = 0$ implies that a and b are the roots of $x^2+4x+2 = 0$ which are real. By Viete's Relations, $ab = 2$. Hence, the answer is B. ■

5. *The function $f(x) = \dfrac{1-x}{x^2+2}$ has a maximum value of $f(a)$ and a minimum value of $f(b)$. Then, what is the value of $a^2 + b^2$?*

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

Solution: We calculate the derivative $f'(x) = \frac{x^2-2x-2}{(x^2+2)^2}$ and observe that a and b are the zeros of $x^2 - 2x - 2 = 0$. One can solve this quadratic equation or observe that $a^2 + b^2 = (a+b)^2 - 2ab = 2^2 - 2(-2) = 8$. Hence, the correct answer is D. ■

6. *The area of the regions between the graphs of equations* $y = 3x$ *and* $y = 2(x^2 - 1)$ *is a rational number which can be written in reduced form as* $\frac{5^m}{3(2^n)}$*, for some integers a and n. What is* $m(n+1)$*?*

(A) 10 (B) 12 (C) 14 (D) 16 (E) 18

Solution: The two curves intersect at points of the form (x, y), where the x's are the solutions of the equation $3x = 2(x^2 - 1)$ or $(2x+1)(x-2) = 0$. This gives $x_1 = -1/2$ and $x_2 = 2$. Hence, the area we are interested in is

$$A := \int_{-1/2}^{2} 3x - 2(x^2 - 1)dx = 3\frac{x^2}{2}|_{-1/2}^{2} - 2\frac{x^3}{3}|_{-1/2}^{2} + 2x|_{-1/2}^{2} \Rightarrow$$

$$A = \frac{3}{2}(\frac{15}{4}) - \frac{2}{3}(\frac{65}{8}) + 2(\frac{5}{2}) = \frac{5}{(3)8}(27 - 26 + 24) = \frac{5^3}{3(2^3)}.$$

This shows that $m = n = 3$. Thus, the correct answer is B. ■

7. *Suppose that* f *is differentiable and* $g(x) = x^2 f(1/x)$*. If* $f(1) = 3$ *and* $f'(1) = 4$*, what is* $g'(1)$*?*

(A) -2 (B) -1 (C) 0 (D) 1 (E) 2

Solution: Using the product rule and the chain rule, we get $g'(x) = 2xf(1/x) + x^2 f'(1/x)(-1/x^2)$ or

$$g'(x) = 2xf(1/x) - f'(1/x),$$

which attracts $g'(1) = 2f(1) - f'(1) = 2(3) - 4 = 2$. Thus, the answer is E. ■

8. *What is the value of the limit*

$$L := \lim_{x\to\infty} \frac{\int_0^x \sqrt{3 + 16t^2}dt}{x^2 + 1}.$$

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: We use L'Hospital's Rule and the Fundamental Theorem of calculus for this, to simplify the given limit in the following way

$$L := \lim_{x\to\infty} \frac{\sqrt{3+16x^2}}{2x} = \frac{1}{2}\lim_{x\to\infty}\sqrt{\frac{3}{x^2}+16} = 2.$$

This gives the answer B. ∎

9. *The positive integers m and n are relatively prime, and chosen in such a way that*

$$L := \lim_{x\to 0} \frac{3\sqrt{49+x} - 7\sqrt{9-x}}{x} = \frac{m}{n}.$$

What is $m+n$?

(A) 20 (B) 30 (C) 40 (D) 50 (E) 60

Solution: We can use L'Hospital's Rule as before, to obtain

$$L = \lim_{x\to 0} 3\frac{1}{2\sqrt{49+x}} + 7\frac{1}{2\sqrt{9-x}} = \frac{1}{2}(\frac{3}{7}+\frac{7}{3}) = \frac{29}{21}.$$

This shows that $m = 29$ and $n = 21$. Hence, $m+n = 50$ and so, the answer is D. ∎

10. *We let m be the smallest positive integer such that*

$$I := m\int_0^2 \frac{1-x}{\sqrt{1+4x}}dx$$

is also an integer. What is m?

(A) 5 (B) 6 (C) 3 (D) 4 (E) 0

Solution: We make the substitution $1+4x = u^2$. This gives $x = \frac{u^2-1}{4}$, and $4dx = 2udu$ or $dx = \frac{1}{2}udu$. Then, we can rewrite I as

$$I = m\int_1^3 \frac{1-\frac{u^2-1}{4}}{u}\frac{1}{2}udu = \frac{m}{8}\int_1^3 (5-u^2)du \Rightarrow$$

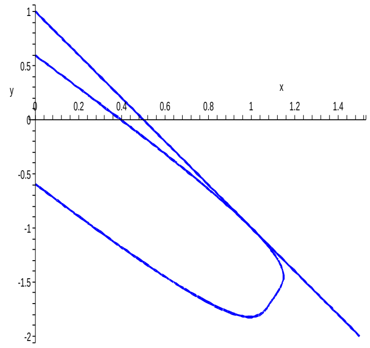
$$I = \frac{m}{8}(10 - \frac{26}{3}) = \frac{m}{6}.$$

Therefore, the answer is B. ∎

11. *Determine the equation of the tangent line to the graph of equation* $(3x + 2y)^4 - x^2 y = 2$ *at the point* $(1, -1)$.

(A) $y = 2x - 3$ (B) $y = 1 - 2x$ (C) $3y + 4x = 1$

(D) $y = -x$ (E) $y = x - 2$

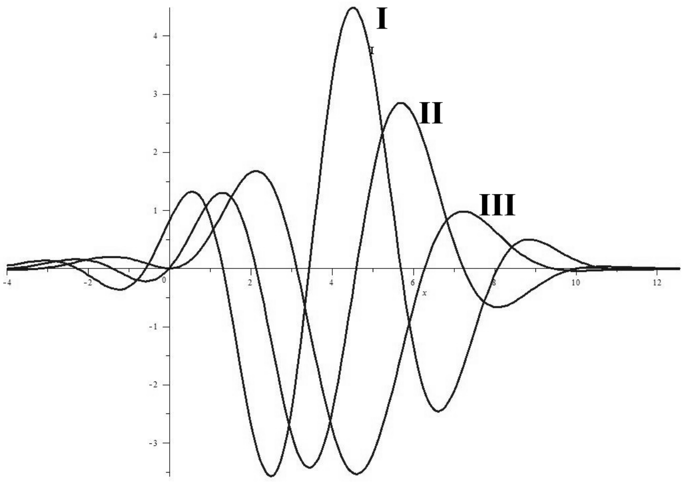


Solution: We use implicit differentiation to get $4(3x+2y)^3(3+2y') - 2xy - x^2y' = 0$. Substituting $x = 1$ and $y = -1$ gives $4(3 + 2y') + 2 - y' = 0$. Solving for y', we obtain $y' = -2$. Hence the equation of the tangent line is $y = -1 - 2(x - 1)$ or $\boxed{y = 1 - 2x}$. Hence, the answer is B. ■

12. *In the accompanying figure we have the graphs of* f, f' *and* f''. *Identify these graphs with the roman numerals shown.*

(A) $I = f$, $II = f'$ (B) $I = f$, $III = f'$ (C) $II = f$, $III = f'$

(D) $II = f$, $I = f'$ (E) $III = f$, $II = f'$



Solution: Where $f' > 0$ the function f should be increasing. This excludes A, B, and

C. In the case of D, if it was correct, then III must be the graph of f'', but then since B is incorrect, it remains that E is the only option. We see that all the relationships are shown to be correct in this case. ■

13. *If $f(x) = x^{x^2}$, then its derivative satisfies*

$$f'(x) = xf(x)(m + n \ln x), \text{ for all } x > 0,$$

with m and n some positive real numbers. What is $m + n$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: We get $\ln(f(x)) = x^2 \ln x$ and so $\frac{f'(x)}{f(x)} = 2x \ln x + x^2 \frac{1}{x}$. This shows that $f'(x) = xf(x)(x + 2 \ln x)$. Then, we must have $m = 1$ and $n = 2$. This shows that the correct answer is C. ■

14. *For m and n relatively prime positive integers, we have*

$$\lim_{x \to 0} (\cos x)^{\cot^2 x} = e^{-m/n}.$$

What is $n - m$.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: The limit is equivalent to $\lim_{x \to 0} \cot^2 x \ln \cos x = -m/n$ or

$$\lim_{x \to 0} \frac{\ln \cos x}{x^2} \frac{x^2}{\sin^2 x} \cos^2 x = -\frac{m}{n} \tag{1}$$

We know that $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0} \cos x = 1$. Hence, if we use L'Hospital's Rule we can rewrite (1) as:

$$\lim_{x \to 0} \frac{-\sin x}{\cos x} \frac{1}{2x} = -\frac{1}{2},$$

which means $m = 1$ and $n = 2$. This implies the answer is A. ■

15. *Find $F'(1)$ for the function $F(x) = \int_{1/x^5}^{x^5} \frac{1}{1 + t^4} dt$, $x \in (0, 2)$.*

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: Using the Fundamental Theorem of Calculus we get $F'(x) = \frac{5x^4}{1+x^{20}} - \frac{-5x^{-6}}{1+x^{-20}}$. This gives $F'(1) = \frac{5}{2} - \frac{-5}{2} = 5$ and so, E is the correct answer. ■

16. Let f be a continuous function defined on $[-1,1]$ and such that $f(x)+f(-x) = x^4$ for all $x \in [-1,1]$. What is $\int_{-1}^{1} f(x)dx$?

(A) 1/2 (B) 1/3 (C) 1/4 (D) 1/5 (E) 1/6

Solution: We can integrate the identity $f(x) + f(-x) = x^4$ on the interval $[-1,1]$ to obtain

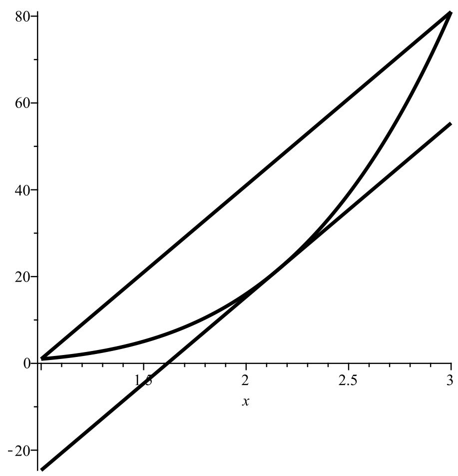
$$\int_{-1}^{1} f(x)dx + \int_{-1}^{1} f(-x)dx = \int_{-1}^{1} x^4 dx = \frac{x^5}{5}|_{1}^{1} = \frac{2}{5}.$$

But, if we make the substitution $-x = u$ in the second integral we get $\int_{-1}^{1} f(-x)dx = \int_{1}^{-1} f(u)(-du) = \int_{-1}^{1} f(u)du$. Hence $2\int_{-1}^{1} f(x)dx = \frac{2}{5}$, which gives D as the correct answer. ■

17. [$*^4$] *We let $A(1,1)$ and $B(3,81)$ be two points on the graph of $y = x^4$. We consider a point $C(c,c^4)$ on the same graph and in between A and B, such that the triangle ABC has the greatest area. What is then c^3?*

(A) 8 (B) 10 (C) 12

(D) 14 (E) 16



Solution: We observe that the triangle ABC has a maximum area when the parallel through C to $\overline{AB}$, is actually tangent to the graph $y = x^4$. If it is not tangent one can easily see it separates points on the curve and the points A and B which will alow one to find triangles ABC' with bigger area. Since the slope of $\overline{AB}$ is equal to 40 and $dy/dx = 4x^3$ we get the equation $40 = 4c^3$ which implies that B is the correct answer here. ■

18. [$*^3$] *The function E satisfies the differential equation $E'(t) = E(t)^2$ and the initial condition $E(0) = 1$. What is the value of $E(2/3)$?*

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Solution: We observe that $\frac{E'(t)}{E(t)^2} = 1$ for t close to 0. This is equivalent to $\frac{d}{dt}(t + \frac{1}{E(t)}) = 0$, which means the function $t + \frac{1}{E(t)}$ is a constant whenever $E(t)$ is not zero. Because $E(0) = 1$ we get that $t + \frac{1}{E(t)} = 1$ which forces $E(t) = \frac{1}{1-t}$ for all $t < 1$. Therefore, $E(3/4) = 4$ and so E is the answer here. ■

19. [$*^2$] *Find the length of the curve given by* $y = \arcsin(x) + \sqrt{1-x^2}$, $x \in [-1, 1]$.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: Using the arc-length formula we get

$$L = \int_{-1}^{1} \sqrt{1 + (dy/dx)^2} dx.$$

Since $dy/dx = \frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}$ we see that $1 + (dy/dx)^2 = 1 + \frac{1-x}{1+x} = \frac{2}{1+x}$. This means we can do a substitution, such as $1 + x = u^2$ and get

$$L = \sqrt{2} \int_{0}^{\sqrt{2}} \frac{1}{u} 2u du = 4.$$

Thus, D is the correct answer. ■

20. [$*^1$] Find the limit

$$\lim_{n\to\infty} \frac{1}{n}(2^{1/n} + 2^{2/n} + \cdots + 2^{n/n}).$$

(A) $\sqrt[3]{2}$ (B) $1 + \frac{\ln 2}{2}$ (C) $2 \ln 2$ (D) $\sqrt{2}$ (E) $\frac{1}{\ln 2}$

Solution: We use the Riemann Sums definition of the definite integral and see that the limit is equal to

$$\int_{0}^{1} 2^x dx = \frac{2^x}{\ln 2}\Big|_0^1 = \frac{1}{\ln 2},$$

and so E is the answer here. ■