First Annual Columbus State Pre-Calculus Tournament

Sponsored by
The Columbus State University
Department of Mathematics
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The Columbus State University Mathematics faculty welcome you to this year's Pre-Calculus/Calculus tournament. We wish you success on this test and in your future studies.

Instructions

This is a 120-minute, 10-problem or 20-problem, multiple choice examination. There are five possible responses to each question. You should select the one "*best*" answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with a No. 2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response.

Throughout the exam, $\overline{AB}$ will denote the line segment from point A to point B and AB will denote the length of $\overline{AB}$. Pre-drawn geometric figures are not necessarily drawn to scale. The measure of the angle $\angle ABC$ is denoted by $m\angle ABC$.

The examination will be scored on the basis of $+12$ *for each correct answer,* -3 *for each incorrect selection, and* $+1$ *for each omitted item.*

No phones or any communication devices can be used. Calculators with CAS such as the TI-89 are not allowed. In fact, the test is designed in such a way that you do not really need a calculator. The problems denoted with $[\star^n]$ are the tie-breaker problems, so more attention should be given to those in the spare time and please add written justification for your answers on extra paper for these problems in the order $[\star^1]$, $[\star^2]$, etc. It is not really necessary, but you may find useful to read the "Theoretical facts" part.

Do not open your test until instructed to do so!

Theoretical facts that you may find useful.

Theorem 1: (Factor Theorem) Given a polynomial P, then P is divisible by $x-a$ if and only if $P(a)=0$. The remainder of the division of $P(x)$ by $x-a$ is $P(a)$.

Theorem 2: The trigonometric addition/subtraction formulae:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Theorem 3: A polynomial $P(x) = a_0 + a_1x + \cdots a_nx^n$ has a rational zero, p/q (reduced form), if p divides a_n and q divides a_0.

Theorem 4: Quadratic formula and Viete's relations: the equation $ax^2+bx+c=0$ has two zeros (x_1, x_2, real or complex) given by

$$\boxed{x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}}.$$

These zeros satisfy Viete's relations:

$$\boxed{x_1 + x_2 = -\frac{b}{a}, \quad x_1x_2 = \frac{c}{a}}.$$

Theorem 5: Heron's formula: The area of a triangle is given by the formula

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{a+b+c}{2}$ and a, b and c are the sides of the triangle.

Theorem 6: The Law of Sines in a triangle states that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin A}$$

where A, B and C are the measurement of the interior angles of the triangle.

Theorem 7: (Pythagorean Theorem:) In a right triangle with legs b, c and hypothenuse a, we have $b^2+c^2=a^2$.

Theorem 8: Properties of logarithms:

$$\log_a u = v \Leftrightarrow u = a^v$$

$$\log_a u + \log_a v = \log_a uv, \quad \log_a u^v = v\log_a u$$

Precalculus Problems

1. Find the remainder of the division $(x^4 - 4x^3 + 6x^2 - 3x + 1) \div (x^2 - 2x + 1)$

   (A) $2x - 1$ (B) x (C) $3x - 2$ (D) $2 - x$ (E) $x - 1$

2. For some positive numbers a and b we have the identity

$$\frac{\sin 9x}{\cos 3x} + \frac{\cos 9x}{\sin 3x} = a \cot bx,\ \ x \in (0, \frac{\pi}{12}).$$

   What is $2a + b$?

   (A) 9 (B) 10 (C) 11 (D) 12 (E) 13

3. If θ is an angle in the third quadrant and $\tan\theta = \frac{28}{45}$, what is the value of $28 \csc\theta$?

   (A) -28 (B) -45 (C) 45 (D) 53 (E) -53

4. $[*^5]$ The cubic equation $2x^3 + 3x^2 + 5x + 2 = 0$ has two solutions, x_1 and x_2, which are not real. Find $x_1 + x_2$.

   (A) -1 (B) 1 (C) -2 (D) 2 (E) -3

5. Find the area of the triangle with sides $a = 9$, $b = 10$ and $c = 17$.

   (A) 30 (B) 32 (C) 34 (D) 36 (E) 38

6. $[*^4]$ If $t = \log_4 a = \log_6 b = \log_9(a - \frac{3}{2}b)$, what is $\frac{a}{b}$?

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

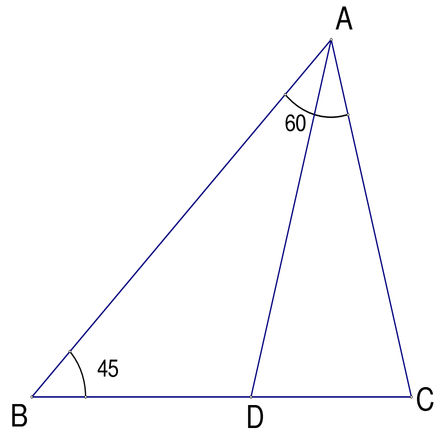
7. The equation $x^{\log_3 x} = \frac{x^3}{9}$ has two solutions, say x_1 and x_2, with $x_1 < x_2$. What is $x_2 - x_1$? Hint: Consider the logarithm base three of each side of the given equation.

(A) 2 (B) 4 (C) 6 (D) 8 (E) 10

8. [$*^3$] In the triangle ABC the angle $\angle A$ is $60°$ and the angle $\angle B$ is equal to $45°$. The angle bisector of the angle $\angle A$ intersects $\overline{BC}$ at D. Knowing that $AD = 10$ and that $BC = m\sqrt{n}$, where m and n are natural numbers with n not divisible by the square of a prime number, what is $m + n$?

(A) 19 (B) 15 (C) 11

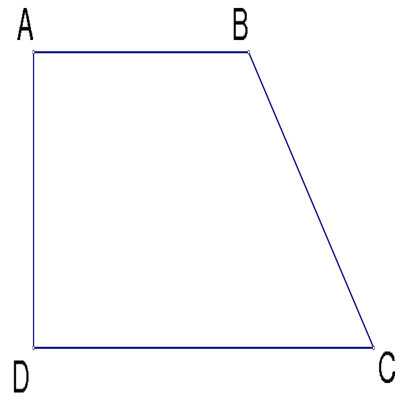
(D) 7 (E) 3



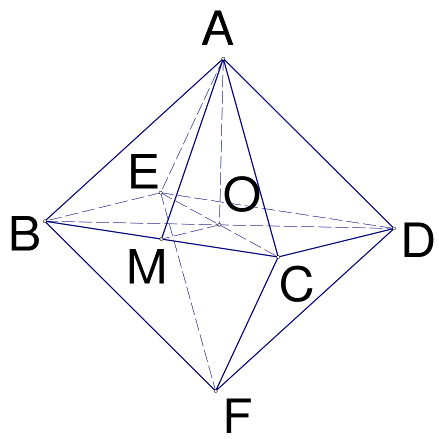
9. [$*^2$] In the trapezoid $ABCD$, $\overline{AB}$ and $\overline{CD}$ are perpendicular to $\overline{AD}$. Knowing that $AD = 6 < BC = AB + CD$, what is $AB \cdot CD$?

(A) 9 (B) 8 (C) 12

(D) 16 (E) 6



10. [$*^1$] In the accompanying figure we have a regular octahedron. We denote the dihedral angle between its faces by α (that is, $\alpha = 2(m\angle AMO)$). What is the value of $\cos\alpha$?



(A) $-\frac{1}{2}$ (B) $-\frac{1}{3}$ (C) $-\frac{2}{3}$

(D) $-\frac{1}{4}$ (E) $-\frac{3}{4}$

End of Pre-Calculus Problems