Solutions to the Second Annual Columbus State Calculus Contest

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1. The function $f(x) = \dfrac{x+1}{x^2+3}$ has a maximum value of $f(a)$ and a minimum value of $f(b)$. Then, what is the value of $a - 3b$?

   (A) 7 (B) 8 (C) 9 (D) 10 (E) 11

   Solution: We calculate the derivative, $f'(x) = \frac{(x+3)(1-x)}{(x^2+3)^2}$, and observe that $a = 1$ and $b = -3$. Hence $a - 3b = 10$ and the correct answer is D. ■

2. Suppose that f is differentiable and $g(x) = f(\frac{1+3x}{1-5x})$ for all $x \neq \frac{1}{5}$. If $f'(-1) = 2$, what is $g'(1)$?

   (A) -2 (B) -1 (C) 0 (D) 1 (E) 2

   Solution: Using the chain rule and the quotient rule, we get

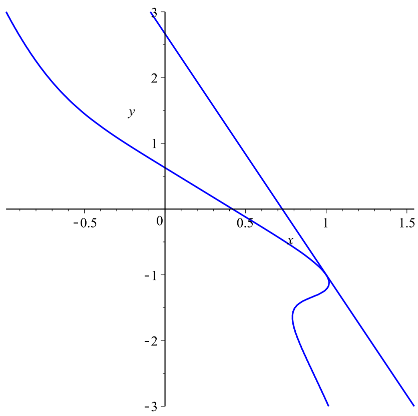
$$g'(x) = f'(\frac{1+3x}{1-5x})\frac{3(1-5x)-(1+3x)(-5)}{(1-5x)^2} = f'(\frac{1+3x}{1-5x})\frac{8}{(1-5x)^2}$$

   Then $g'(1) = f'(-1)\frac{8}{(-4)^2} = 2\frac{8}{16} = 1$. Then, the answer is D. ■

3. Determine the equation of the tangent line to the graph of equation $(3x+2y)^3 - x^2y^3 = 2$ at the point $(1, -1)$.

   (A) $2x - y - 3 = 0$ (B) $11x + 3y = 8$ (C) $11x + 3y = 14$

   (D) $4x + y = 3$ (E) $5x + y = 4$



Solution: We use implicit differentiation to get $3(3x+2y)^3(3+2y')-(2xy^3+x^2 3y^2 y') = 0$. Substituting $x = 1$ and $y = -1$ gives $3(3+2y')-(-2+3y') = 0$. Solving for y', we obtain $y' = -11/3$. Hence the equation of the tangent line is $y = -1 - 11(x-1)/3$ or $\boxed{11x + 3y = 8}$. Hence, the answer is B. ■

4. (I) There are continuous functions at a point which are not differentiable at that point.

   (II) Every continuous function has the Intermediate Value Property.

   (III) There are differentiable functions which are not continuous.

   Which of the choices below best describes the subset of above true statements?

   (A) (I) (B) (II) (C) (III)

   (D) (III) and (IV) (E) (I) and (II)

Solution: (I) is true since, for instance, the function $f(x) = |x|$ is continuous at 0 but not differentiable at this point. (II) is true as we have stated in the Intermediate Value Theorem. (III) is false since

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

implies

$$\lim_{x\to a} f(x) = f(a).$$

Hence, the answer is E. ■

5. On what interval $[a, b]$ can we apply the Intermediate Value Theorem to the function $f(x) = 3\sin x - 2\cos x$ for x in $[a, b]$, in order to obtain a value c in (a, b) such that $f(c) = 0$?

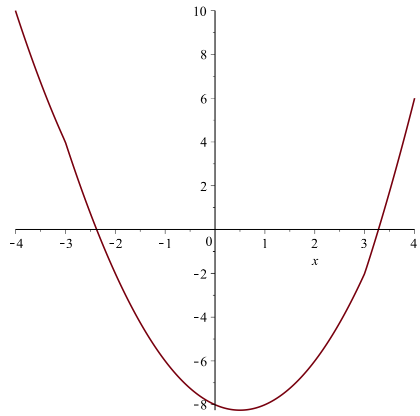
   (A) $[0, 2\pi]$ (B) $[\frac{\pi}{2}, \pi]$ (C) $[\frac{3\pi}{2}, 2\pi]$ (D) $[-\frac{\pi}{2}, 0]$ (E) $[0, \frac{\pi}{2}]$

Solution: In order to apply the IVT, we need to have (in this case) $f(a)f(b) < 0$. For A, we get $f(0)f(2\pi) = (-2)^2 > 0$. For B, we obtain $f(\frac{\pi}{2})f(\pi) = 6 > 0$. Also, $f(3\frac{\pi}{2})f(2\pi) = 6$ and $f(-\frac{\pi}{2})f(0) = 6 > 0$. Finally, we have $f(0)f(\frac{\pi}{2}) = -2(3) = -6 < 0$, which shows that E is the right answer. ■

6. The values of a and b are chosen in such a way the function

$$f(x) = \begin{cases} ax^2 + bx - 2 \text{ if } x \leq -3, \\ x^2 - ax - 8 \text{ if } -3 < x < 3, \\ bx^2 + ax - 14 \text{ if } x \geq 3, \end{cases}$$

is continuous on the whole real line (graph as in the adjacent figure). What is $(a+b)^2$?



(A) 0 (B) 1 (C) 4

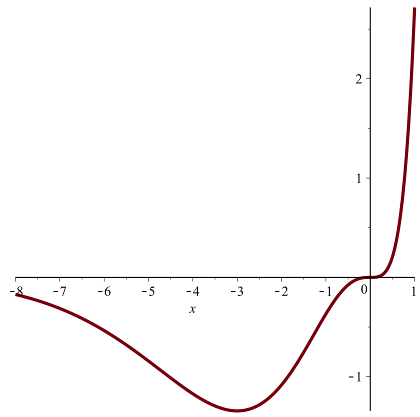
(D) 9 (E) 16

Solution: The function needs to be continuous at -3 which implies $f(-3) = \lim_{x \searrow -3} f(x)$. This is equivalent to $9a - 3b - 2 = 1 + 3a$ or $2a = b + 1$. Similarly, we need to have $f(3) = \lim_{x \nearrow 3} f(x)$ or $9b + 3a - 14 = 1 - 3a$ or $3b + 2a = 5$. This gives $a = b = 1$ and so then $(a + b)^2 = 4$ and therefore the answer is C. ■

7. The function $f(x) = x^3 e^x$ has inflection points at $x = 0$, $x = a$ and $x = b$ with $a < b$. Then what is ab ?

(A) 2 (B) 3 (C) 4

(D) 5 (E) 6

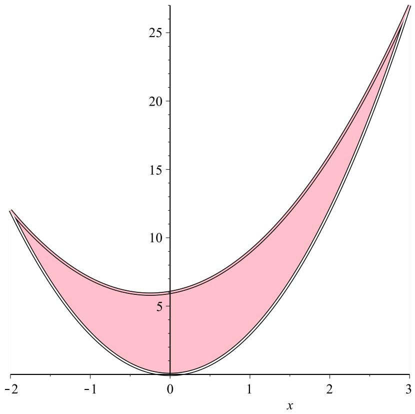


Solution: First, we calculate the derivative $f'(x) = x^2(x + 3)e^x$ and then the second derivative $f''(x) = x(6 + 6x + x^2)e^x$. Then, a and b are the roots of $x^2 + 6x + 6 = 0$. By Viete's Relations, $ab = 6$. Hence, the answer is E. ■

8. The area of the region between the graphs of the parabolas $y = 3x^2$ and $y = 2x^2 + x + 6$ is a rational number which can be written in reduced form as $\frac{25m}{2n}$, for some integers m and n (see the adjacent figure). What is $4n - m$?

   (A) 8 (B) 7 (C) 6

   (D) 5 (E) 4



Solution: The two curves intersect at points of the form (x, y), where the x's are the solutions of the equation $3x^2 = 2x^2 + x + 6$ or $(x - 3)(x + 2) = 0$. This gives $x_1 = -2$ and $x_2 = 3$. Hence, the area we are interested in is

$$A := \int_{-2}^{3} (x + 6 - x^2)dx = \frac{x^2}{2}|_{-2}^{3} + 6x|_{-2}^{3} - \frac{x^3}{3}|_{-2}^{3} \Rightarrow$$

$$A = \frac{5}{2} + 30 - \frac{35}{3} = \frac{15 + 180 - 70}{3} = \frac{125}{3}.$$

This shows that $m = 5$ and $n = 3$. Thus, $4n - m = 7$ and so the correct answer is B.■

9. What is the value of the limit

$$\lim_{x\to\infty} \frac{\int_0^{x^2} \frac{t^4}{1+t^3} dt}{x^4}.$$

   (A) 1 (B) 1/2 (C) 1/3 (D) 1/4 (E) 1/5

Solution: We use L'Hospital's Rule and the Fundamental Theorem of calculus for this, to simplify the given limit in the following way

$$L := \lim_{x\to\infty} \frac{\frac{x^8}{1+x^6} 2x}{4x^3} = \frac{1}{2} \lim_{x\to\infty} \frac{x^9}{x^9 + x^3} = \frac{1}{2}.$$

This gives the answer B. ■

10. The positive integers m and n are relatively prime, and chosen in such a way that

$$\lim_{x\to 2}\frac{5\sqrt{1+4x}-3\sqrt{1+12x}}{x-2}=-\frac{m}{n}.$$

What is $n-4m$?

(A) -1 (B) 2 (C) -3 (D) 4 (E) -5

Solution: We can use L'Hospital's Rule as before, to obtain

$$L=\lim_{x\to 2}20\frac{1}{2\sqrt{1+4x}}-36\frac{1}{2\sqrt{1+12x}}=\frac{10}{3}-\frac{18}{5}=-\frac{4}{15}.$$

This shows that $m=4$ and $n=15$. Hence, $n-4m=-1$ and so, the answer is A. ■

11. We let m be the smallest positive integer such that

$$m\left(\frac{2\ln 2}{3}-\int_0^1 x^2\ln(x+1)dx\right)$$

is also an integer. What is m?

(A) 15 (B) 16 (C) 17 (D) 18 (E) 19

Solution: We make the substitution $x+1=e^t$. This gives

$$I:=\int_0^1 x^2\ln(x+1)dx=\int_0^{\ln 2}(e^t-1)^2te^tdt=\int_0^{\ln 2}(te^{3t}-2te^{2t}+te^t)dt.$$

We can use the formula $\int P(x)e^{ax}dx=e^{ax}(P(x)/a-P'(x)/a^2+...)+C$. We can then continue

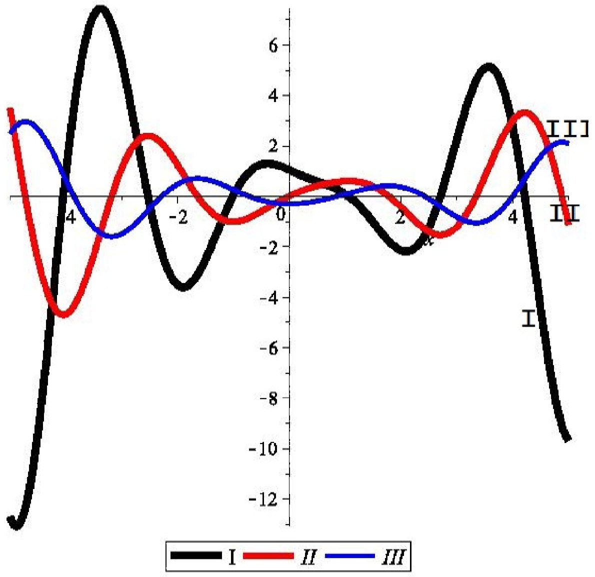
$$I=e^{3t}(\frac{t}{3}-\frac{1}{9})|_0^{\ln 2}-2e^{2t}(\frac{t}{2}-\frac{1}{4})|_0^{\ln 2}+e^t(t-1)|_0^{\ln 2}$$

or, $I=\ln 2(\frac{8}{3}-2(4)\frac{1}{2}+2)-\frac{8}{9}+\frac{1}{9}+2-\frac{1}{2}-2+1=\frac{2\ln 2}{3}-\frac{5}{18}$.
Therefore, then $m=18$ and so the answer is D. ■

12. In the accompanying figure we have the graphs of f, f' and f''. Identify these graphs with the roman numerals shown.

(A) $I=f$, $II=f'$ (B) $I=f$, $III=f'$ (C) $II=f$, $III=f'$

(D) $II=f$, $I=f'$ (E) $III=f$, $II=f'$



Solution: Where $f' > 0$ the function f should be increasing. This excludes A, B, and C. In the case of D, if it was correct, then III must be the graph of f'', but then since B is incorrect, it remains that E is the only option. We see that all the relationships are shown to be correct in this case. ■

13. If $f(x) = x^{\sqrt{x}}$, then its derivative satisfies

$$f'(x) = \frac{1}{2}x^{\sqrt{x}-\frac{1}{2}}(m + n\ln x), \text{ for all } x > 0,$$

with m and n some positive real numbers. What is $m + 3n$?

(A) 5 (B) 4 (C) 3 (D) 2 (E) 1

Solution: We get $\ln(f(x)) = \sqrt{x}\ln x$ and so $\frac{f'(x)}{f(x)} = \frac{1}{2\sqrt{x}}\ln x + \sqrt{x}\frac{1}{x}$. This shows that $f'(x) = \frac{1}{2}x^{\sqrt{x}-\frac{1}{2}}(2 + \ln x)$. Then, we must have $m = 2$ and $n = 1$. This shows that the correct answer is A. ■

14. For m and n relatively prime positive integers, we have

$$\lim_{x\to\frac{\pi}{2}}\left(\tan\frac{x}{2}\right)^{\tan x} = e^{-m/n}.$$

What is $n - m$.

(A) 1 (B) 0 (C) -2 (D) -3 (E) -4

Solution: The limit is equivalent to $\lim_{x\to\frac{\pi}{2}} \tan x \ln\tan\frac{x}{2} = -m/n$ or

$$\lim_{x\to\frac{\pi}{2}} \frac{\ln\tan\frac{x}{2}}{\cos x}\sin x = -\frac{m}{n} \tag{1}$$

We know that $\lim_{x\to\frac{\pi}{2}} \sin x = 1$, $\lim_{x\to\frac{\pi}{2}} \cos x = 0$ and $\lim_{x\to\frac{\pi}{2}} \ln\tan\frac{x}{2} = 0$. Hence, if we use L'Hospital's Rule we can rewrite (1) as:

$$\lim_{x\to\frac{\pi}{2}} \frac{\frac{\sec^2(x/2)}{2\tan(x/2)}}{-\sin x} = -1,$$

which means $m = n = 1$. This implies the answer is B. ■

15. Find $F'(1)$ for the function $F(x) = \displaystyle\int_x^{x^3} \frac{1}{t+t^5}dt$, for x in $(0, 2)$.

    (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: Using the Fundamental Theorem of Calculus we get $F'(x) = \frac{3x^2}{x^3+x^{15}} - \frac{1}{x+x^5}$. This gives $F'(1) = \frac{3}{2} - \frac{1}{2} = 1$ and so, A is the correct answer. ■

16. Let f be a continuous function defined on $[0, \pi]$ and such that $f(x) + f(\pi - x) = \sin x$ for all $x \in [0, \pi]$. What is $\int_0^\pi f(x)dx$?

    (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: We can integrate the identity $f(x) + f(-x) = \sin x$ on the interval $[0, \pi]$ to obtain

$$\int_0^\pi f(x)dx + \int_0^\pi f(\pi - x)dx = \int_0^\pi \sin x dx = -\cos x|_0^\pi = 2.$$

But, if we make the substitution $\pi - x = u$ in the second integral we get $\int_0^\pi f(-x)dx = \int_\pi^0 f(u)(-du) = \int_0^\pi f(u)du$. Hence $2\int_0^\pi f(x)dx = 1$, which gives A as the correct answer. ■

17. [$*^4$] We let $A(1, 1)$ and $B(2, 8)$ be two points on the graph of $y = x^3$. We consider a point $C(c, c^3)$ on the same graph and in between A and B, such that the triangle ABC has the greatest area. What is the value of c^2?

    (A) 8/3 (B) 10/3 (C) 2 (D) 7/3 (E) 5/3

Solution: We observe that the triangle ABC has a maximum area when the parallel through C to $\overline{AB}$, is actually tangent to the graph $y = x^3$. If this parallel is not tangent to the graph, one can easily see it separates points on the curve and the points A and B which will alow one to find triangles ABC' with bigger area. Since the slope of $\overline{AB}$ is equal to 7 and $dy/dx = 3x^2$ we get the equation $7 = 3c^2$ which implies that D is the correct answer here. ■

18. [$*^3$] The function E satisfies the differential equation $E'(t) = -E(t)^3$ and the initial condition $E(0) = 1$. What is the value of $E(4)$?

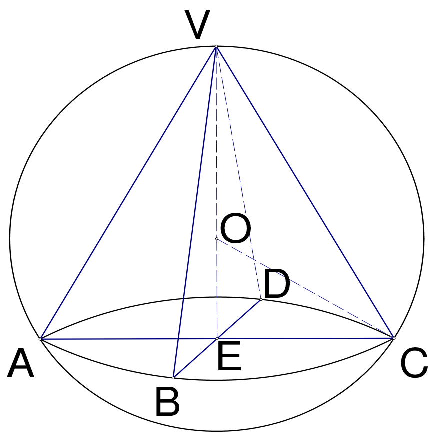
(A) 0 (B) 1/3 (C) 2/3 (D) 1 (E) 4/3

Solution: We observe that $\frac{E'(t)}{E(t)^3} = -1$ for t close to 0. This is equivalent to $\frac{d}{dt}(t - \frac{1}{2E(t)^2}) = 0$, which means the function $t - \frac{1}{2E(t)}$ is a constant whenever $E(t)$ is not zero. Because $E(0) = 1$ we get that $t - \frac{1}{2E(t)^2} = -1/2$ which forces $E(t) = \frac{1}{\sqrt{2t+1}}$ for all $t > -1/2$. Therefore, $E(4) = 1/3$ and so B is the answer here. ■

19. [$*^2$] A regular square pyramid is inscribed in a sphere of radius R. What is the maximum volume of such a pyramid?

(A) $\frac{16R^3}{27}$ (B) $\frac{32R^3}{25}$ (C) $\frac{32R^3}{81}$

(D) $\frac{16R^3}{27}$ (E) $\frac{64R^3}{81}$



Solution: In the above figure we have $VABCD$ the regular square pyramid with side lengths of the base $AB = BC = CD = AD = x$ and height $VE = h$. This gives the optimization function $V(x) = \frac{x^2h}{3}$. Because of the symmetry we may assume that the center of the sphere is on $\overline{VE}$ and so, since $EC = \frac{x}{\sqrt{2}}$ we have $h = R + OE = R + \sqrt{R^2 - x^2/2}$. Taking the derivative of V and solving for critical points we find $x = 4R/3$. This gives $V(4R/3) = 64R^3/81$ and the correct answer E.

20. [$*^1$] Find the limit

$$L := \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{4n^2 - k^2}}.$$

(A) $\frac{\pi}{5}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{3}$ (E) $\frac{\pi}{2}$

Solution: We use the Riemann Sums definition of the definite integral and see that the limit is equal to

$$L = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2n} \frac{1}{\sqrt{1 - k^2/4n^2}} = \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} dx = \arcsin x|_0^{\frac{1}{2}} = \arcsin \frac{1}{2} = \frac{\pi}{6},$$

and so C is the answer here. ■