Solutions to the Third Annual Columbus State Calculus Contest (Version I)

Sponsored by

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1. Suppose that f is differentiable and defined on $\mathbb{R}$, $g(x) = \ln(x + \sqrt{x^2+1})$ for all real values of x, and $h = f \circ g$. If $f'(\ln 2) = 5$, what is $h'(\frac{3}{4})$?

   (A) 1 (B) 2 (C) 3 (D) $\boxed{4}$ (E) 5

   Solution: Using the chain rule, we get $h' = (f' \circ g)g'$. But we know that $g'(x) = 1/\sqrt{x^2+1}$. Hence,
$$h'(x) = f'(\ln(x + \sqrt{x^2+1}))\frac{1}{\sqrt{x^2+1}}.$$
   Then $h'(3/4) = f'(\ln 2)\frac{4}{5} = 4$. Ergo, the answer is D. ■

2. The function $f(x) = \dfrac{x+3}{x^2+7}$ defined on the whole real line, has a maximum value of $f(a)$ and a minimum value of $f(b)$. Then, what is the value of $5a + b$?

   (A) -1 (B) $\boxed{-2}$ (C) -3 (D) -4 (E) -5

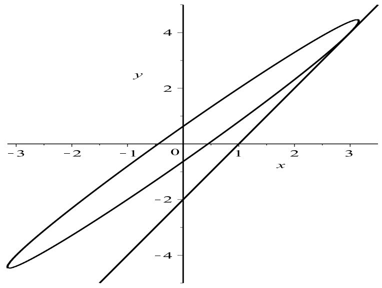
   Solution: We calculate the derivative, $f'(x) = \frac{(x+7)(1-x)}{(x^2+7)^2}$, and observe that $a = 1$ and $b = -7$. Hence $5a + b = -2$ and the correct answer is B. ■

3. If for all real x we define $f(x) = e^{4x} - 8e^{2x} + 16x$, then the inverse of f, f^{-1}, exists and it is differentiable. Calculate $(f^{-1})'(-7) = \frac{d}{dx}(f^{-1})(-7)$.

   (A) 1 (B) 1/2 (C) 1/3 (D) $\boxed{1/4}$ (E) 1/5

   Solution: Let us calculate the derivative of f: $f'(x) = 4e^{4x} - 16e^{2x} + 16 = 4(e^{2x} - 2)^2 \geq 0$. Since $(f^{-1} \circ f)(x) = x$, we can differentiate this equality with respect to x and substitute $x = 0$ in what we get: $(f^{-1})'(f(0))f'(0) = 1$. This implies $(f^{-1})'(-7)4 = 1$, so the answer is D. ■

4. The equation of the tangent line to the graph of equation $(3x-2y)^2+(x-y)^2 = 2$ at the point $(3,4)$ is $y = mx+n$. What is $2m+n$?



(A) 1 (B) $\boxed{2}$ (C) 3

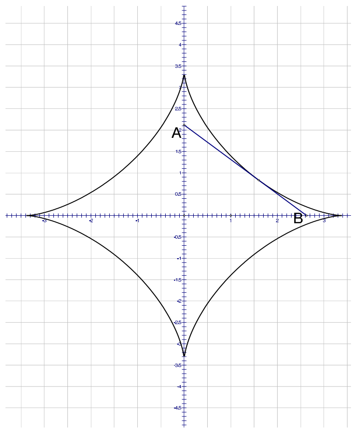
(D) 4 (E) 5

Solution: We use implicit differentiation to get $2(3x-2y)(3-2y')+2(x-y)(1-y') = 0$. Substituting $x = 3$ and $y = 4$ gives $2(3-2y')-2(1-y') = 0$. Solving for y', we obtain $y' = 2$. Hence the equation of the tangent line is $y = 4+2(x-3)$ or $\boxed{y = 2x-2}$. Hence, the answer is B. ■

5. The astroid

$$4(x^{\frac{2}{3}}+y^{\frac{2}{3}}) = 9$$

has the property that each segment $\overline{AB}$ in the first quadrant, with A on the y-axis and B on the x-axis and tangent to the astroid, has the same length L. Find $\lfloor L \rfloor$ (the integer part of L) in this case.



(A) 1 (B) 2 (C) $\boxed{3}$

(D) 4 (E) 5

Solution: To make it easier with the intuition we refer to the figure above. Since the property stated holds true for the limiting position when $\overline{AB}$ becomes horizontal, we can calculate easily $L = x$, when $y = 0$. Solving for x in this case, $x = L = \frac{27}{8} = 3+\frac{3}{8}$. So, $\lfloor L \rfloor = 3$ which makes C the correct answer. ■

6. The function $F(x) = -x+x\ln(x)$ is defined for every $x > 0$. For every natural number n, The Mean Value Theorem applied to F on the interval $[a,b] = [e^n, e^{n+1}]$ gives $F(b)-F(a) = (b-a)F'(c)$ with $c \in (a,b)$. If $c = e^{n+\alpha}$, then what is $\frac{1}{\alpha}$?

(A) $e-2$ (B) $\boxed{e-1}$ (C) e (D) $e+1$ (E) $e+2$

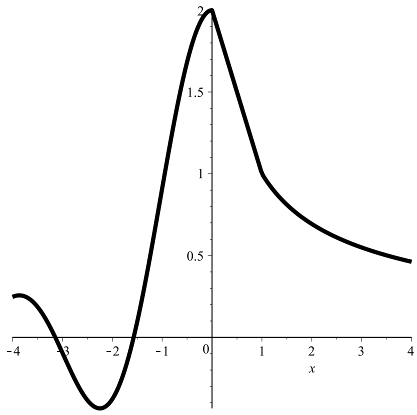
Solution: Since $F'(x) = \ln x$, $F(e^{n+1}) = e^{n+1}n$, and $F(e^n) = e^n(n-1)$, we get $e^{n+1}n - e^n(n-1) = (e^{n+1}-e^n)\ln c$. This can be simplified to $en-n+1 = (e-1)\ln c$

or $c = e^{n+\frac{1}{e-1}}$. Therefore, we see that the correct answer is D. ■

7. The values of a and b are chosen in such a way the function k, piecewisely defined by

$$k(x) = \begin{cases} \dfrac{\sin(2x)}{x} & \text{if } x < 0, \\ ax + b & \text{if } 0 \le x \le 1, \\ \dfrac{\ln(x)}{x-1} & \text{if } x > 1, \end{cases}$$

is continuous on the whole real line (graph as in the adjacent figure). What is $a + 2b$?

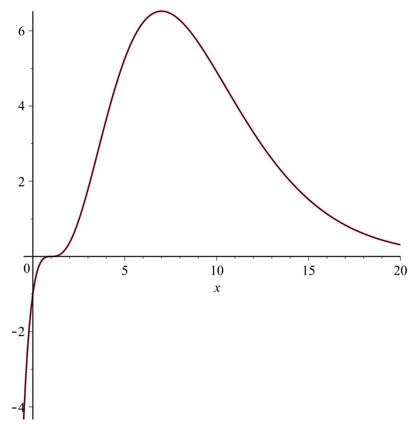


(A) 1 (B) 2 (C) $\boxed{3}$

(D) 4 (E) 5

Solution: Using L'Hospital's Rule we get that $\lim_{x \nearrow 0} k(x) = \lim_{x\to 0} 2\cos(2x) = 2$ which means that $b = 2$. Similarly, $\lim_{x \searrow 1} k(x) = \lim_{x\to 1}(1/x) = 1$, which means $a + b = 1$ and so $a = -1$. Therefore the answer is C. ■

8. The function $f(x) = (x-1)^3 e^{-x/2}$ has inflection points at $x_1 = 1$, $x_2 = a + b\sqrt{3}$ and $x_3 = a - b\sqrt{3}$ with a and b natural numbers. Then what is $a - b$?

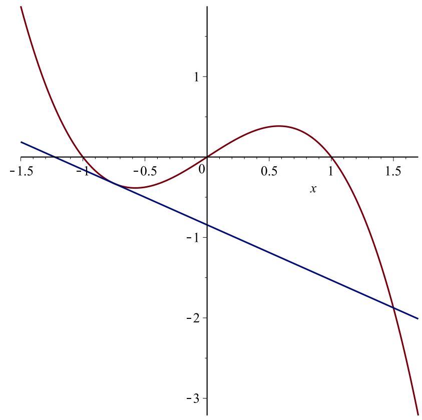


(A) 1 (B) 2 (C) 3

(D) 4 (E) $\boxed{5}$

Solution: First, we calculate the derivative $f'(x) = \frac{1}{2}e^{-x/2}(7-x)(x-1)^2$ and then the second derivative $f''(x) = \frac{1}{4}e^{-x/2}(x-1)(x^2-14x+37)$. This gives $x_{2,3} = 7 \pm \sqrt{49-37} = 7 \pm 2\sqrt{3}$. Hence, the answer is E. ■

9. The cubic of equation $y = h(x) = x - x^3$ is shown in the figure on the right, together with its tangent line at the point $(-\frac{2}{5}, h(-\frac{2}{5}))$. This tangent line intersects the cubic at another point: $(\frac{m}{n}, h(\frac{m}{n}))$, where m and n are relatively prime natural numbers. Find $n - m$.



(A) $\boxed{1}$ (B) 2 (C) 3

(D) 4 (E) 5

Solution: Using the Taylor polynomial to approximate h we get that this is in fact an exact formula:

$$h(x) = h(a) + h'(a)(x-a) + \frac{1}{2}h''(a)(x-a)^2 + \frac{1}{6}h'''(a)(x-a)^3,$$

for any point a. So, if we want to solve the equation $h(x) = h(a) + h'(a)(x-a)$, we obtain equivalently, $x = a$ a double root, and the equation $-3a - (x-a) = 0$ or $x = -2a$. So, in our particular case the tangent line intersects the curve $y = h(x)$ again at $f(\frac{4}{5}, h(\frac{4}{5}))$. Thus, $n - m = 1$ and so the correct answer is A. ■

10. The function $g(x) = \frac{x^{2015}}{1-x}$ is defined for every $x \in (-1, 1)$. If $g^{(n)}$ denotes the n^{th} derivative of g, what is the value of $g^{(2014)}(0)$?

(A) 2014! (B) 2015! (C) 1 (D) -1 (E) $\boxed{0}$

Solution: Using the Taylor series for $\frac{1}{1-x} = 1 + x + x^2 + ...$, we obtain $g(x) = x^{2015} + x^{2016} + ...$, $x \in (-1, 1)$. Then $g'(x) = 2015x^{2014} + ...$ and inductively $g^{(2014)}(x) = 2015!x + (2016!/2!)x^2 + ...$ Therefore, $g^{(2014)}(0) = 0$. This gives the answer E. ■

11. The polynomial function $f(x) = x^4 + mx^3 + 6x^2$, defined for all real values of x, has two distinct inflection points if and only if $|m| > a$. What is the value of a ?

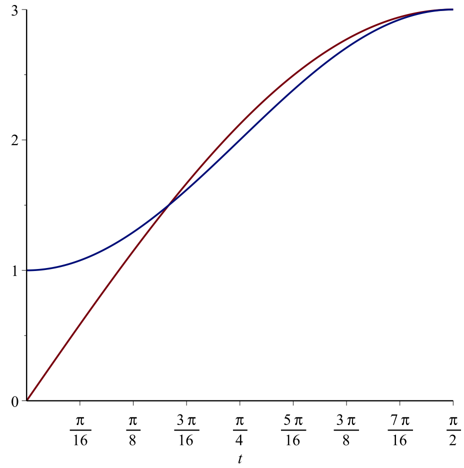
(A) 1 (B) 2 (C) 3 (D) $\boxed{4}$ (E) 5

Solution: We calculate the second derivative and obtain $f''(x) = 12x^2 + 6mx + 12 = 6(2x^2 + mx + 2)$. In order to have two distinct inflection points, we need have two distinct real solutions of the quadratic $2x^2 + mx + 2 = 0$. It is then equivalent to requiring that the determinant $\Delta = m^2 - 4^2 > 0$ or $|m| > 4$. So, the answer is D. ■

12. The graphs of $y = 3\sin x$ and $y = 2 - \cos(2x)$ for $x \in [0, \frac{\pi}{2}]$ are shown in the figure on the right. If A is the area between their graphs in this interval, then $A = \frac{m}{n}\sqrt{3} - \frac{2\pi}{3}$ for a reduced fraction m/n. Calculate $m - n$.

(A) $\boxed{1}$ (B) 2 (C) 3

(D) 4 (E) 5



Solution: We need to see where the two graphs intersect: $3\sin x = 2 - \cos 2x$ or using one of the double angle formulas, this is equivalent to $3\sin x = 2 - (1 - 2\sin^2 x)$. This can be solved by factoring: $(2\sin x - 1)(\sin x - 1) = 0$. So, we get $\sin x = 1/2$ and $\sin x = 1$. Thus, we need to integrate between $\pi/6$ and $\pi/2$. Ergo, $A = \int_{\pi/6}^{\pi/2}(3\sin x - 2 + \cos(2x))dx$ which implies

$$A = -3\cos x|_{\pi/6}^{\pi/2} - 2x|_{\pi/6}^{\pi/2} + \frac{\sin 2x}{2}|_{\pi/6}^{\pi/2} = \frac{5\sqrt{3}}{4} - \frac{2\pi}{3}.$$

This shows that A is the correct answer. ■

13. For m and n relatively prime positive integers, we have

$$\lim_{x\to\pi} |\sin x + \cos x|^{\sec(\frac{x}{2})} = e^{-m/n}.$$

What is $m + 3n$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) $\boxed{5}$

Solution: The limit is equivalent to $\lim_{x\to\pi} \sec(\frac{x}{2}) \ln|\sin x + \cos x| = -m/n$. Using L'Hospital's Rule, we have

$$\lim_{x\to\pi} \frac{\cos x - \sin x}{-\frac{1}{2}\sin(x/2)(\sin x + \cos x)} = -2$$

which means $m = 2$ and $n = 1$. This implies the answer is E. ■

14. Consider the function $F(x) = \int_{\ln x}^{\ln(x+1)} \frac{1}{1+e^{2t}} dt$, for x in $(0, 2)$. Find $-5F'(1)$.

(A) 1 (B) $\boxed{2}$ (C) 3 (D) 4 (E) 5

Solution: Using the Fundamental Theorem of Calculus we get

$$F'(x) = \frac{1}{1+(x+1)^2}\frac{1}{1+x} - \frac{1}{1+x^2}\frac{1}{x}.$$

This gives $F'(1) = \frac{1}{10} - \frac{1}{2} = -\frac{2}{5}$ and so, B is the correct answer. ■

15. We define f by the rule $f(x) = (\sin x)^4 + (\cos x)^4$ for all real numbers x. Knowing that c is the smallest positive number with the property

$$f(c) = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$$

find $\frac{\pi}{c}$.

(A) 12 (B) 10 (C) $\boxed{8}$ (D) 6 (E) 4

Solution: We can simplify f in the following way

$$f(x) = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x = 1 - \frac{1}{2}(\sin 2x)^2 = \frac{3}{4} + \frac{1}{4}\cos(4x).$$

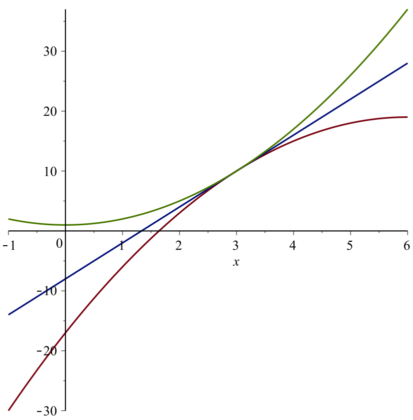
■

So, $\int_0^{2\pi} f(x)dx = \frac{3}{4}(2\pi)$. Then the given equation in c is equivalent to $\cos(4c) = 0$. The smallest positive solution of this equation is clearly given by $4c = \pi/2$, which attracts $\pi/c = 8$. Hence, C is the answer. ■

16. The parabolas $y = x^2 + 1$ and $y = 19 - (x-a)^2$ are tangent to one another and tangent to the line $y = mx + n$ (as in the Figure on the right). Knowing that $a > 0$, find $2m + n$.

(A) 1 (B) 2 (C) 3

(D) $\boxed{4}$ (E) 5



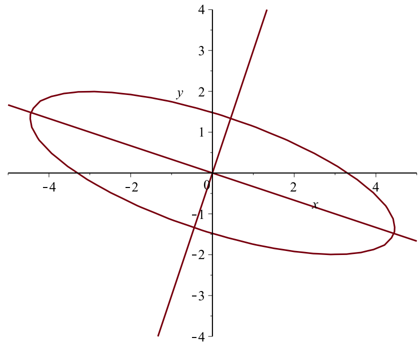
Solution: The common point say (x, y) must satisfy $x^2 + (x-a)^2 = 18$ and $2x =$

$-2(x-a)$. This gives $x = a/2$ and $a^2/4 = 9$. So, $a = \pm 6$. Since $a > 0$ we get $a = 6$. The equation of the tangent line if then $y = 6(x-3) + 10$ or $y = 6x - 8$. Hence the answer is D. ■

17. [$*^1$] What is the smallest value of $x^2 + y^2$ if $x^2 + 3xy + 5y^2 = 11$?

(A) 1 (B) $\boxed{2}$ (C) 3

(D) 4 (E) 5



Solution: We think of a point $(x, y(x))$ on the curve (see figure above) $x^2+3xy+5y^2 = 11$, and as a function of x we need to minimize $x^2 + y(x)^2$. Hence we look for critical points, or $2x + 2yy' = 0$. This implies $y' = -x/y$. Using implicit differentiation we get $2x + 3y + 3xy' + 10yy' = 0$ or $3y^2 - 3x^2 - 8xy = 0$. This homogeneous equation can be solved by factorization $(3y + x)(y - 3x) = 0$. Then $y = 3x$ or $y = -x/3$. So, we need to look where the given curve intersects these two lines. First, we look at $y = 3x$: $x^2 + 3x(3x) + 5(9x^2) = 11$ or $x^2 = \frac{1}{5}$. This gives $x^2 + y^2 = 2$. If we calculate the other intersection points we obtain $x^2 + y^2 = 22$. Therefore, the minimum is 2 and B is the correct answer. ■

18. [$*^2$] The function E satisfies the differential equation $E(t)^3E''(t) = -1$ for every $t \leq 0$ and the initial conditions $E(0) = 1$, $E'(0) = -1$. What is the value of $E(-4)$?

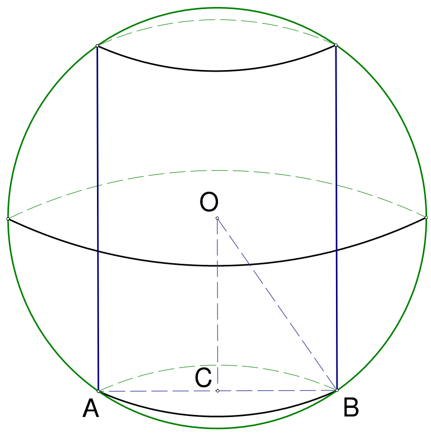
(A) 1 (B) 2 (C) $\boxed{3}$ (D) 4 (E) 5

Solution: Multiplying the equation by $E'(t)$ we get $E''(t)E'(t) = -E(t)^{-3}E'(t)$, which can be integrated to $\frac{1}{2}E'(t)^2 = \frac{1}{2}E^{-2}(t)+C$. Using the initial conditions we get $E'(t) = -E^{-1}(t)$ or $E(t)E'(t) = -1$. We can easily integrate again and obtain $E(t)^2 = -2t+C'$. Then, since $E(0) = 1$ we finally have $E(t) = \sqrt{1-2t}$, $t < 0$. Hence $E(-4) = 3$: answer C. ■

19. [$*^3$] A cylinder is inscribed in a sphere of radius 3 inches as in the adjacent figure. What is the maximum volume of such a cylinder (in cubic inches)?

(A) $4\pi\sqrt{3}$ (B) $6\pi\sqrt{3}$ (C) $8\pi\sqrt{3}$

(D) $10\pi\sqrt{3}$ (E) $12\pi\sqrt{3}$



Solution: Let x be the radius of the cylinder (in the figure above, the segment CB). Then, in terms of R the radius of the sphere, the height of the cylinder is $2OC = 2\sqrt{R^2-x^2}$. Then the volume is given by $V = \pi x^2(2\sqrt{R^2-x^2})$. Maximizing V is equivalent to maximize $V^2 = 4\pi^2x^4(R^2-x^2)$ and so we need to solve for critical points $4R^2x^3 - 6x^5 = 0$. It turns out that V has really a maximum for $x_0 = \frac{2R}{\sqrt{6}}$ ($\frac{d^2V^2}{dx^2}(x_0) < 0$). The maximum is $V(x_0) = 2\pi\frac{R^2}{3}R\frac{\sqrt{3}}{3} = \frac{2\pi R^3\sqrt{3}}{9} = 6\pi\sqrt{3}$: answer B. ■

20. [$*^4$] We denote by L the following limit:

$$L = \lim_{n\to\infty}\left(\frac{n+1}{n}\ln n - \frac{2}{n^2}\sum_{k=1}^{n} k\ln k\right).$$

Find $L/(1-L)$.

(A) $\boxed{1}$ (B) 2 (C) 3 (D) 4 (E) 5

Solution: We use the Riemann Sums definition of the definite integral for $f(x) = x\ln x$ on the interval $(0,1]$. We observe that f, extended at the origin by setting $f(x) = 0$, becomes continuous and so it is Riemann integrable on $[0,1]$. Next, let us observe that, and integration by parts gives,

$$\int_0^1 f(x)dx = \frac{x^2}{2}\ln(x)|_0^1 - \frac{1}{4}x^2|_0^1 = -1/4.$$

Hence

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\ln\frac{k}{n} = -1/4 \Leftrightarrow \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}(\ln(k)-\ln(n)) = -1/4$$

We know that $\sum_{k=1}^{n} k = n(n+1)/2$ and so we can equivalently calculate the above sum:

$$\sum_{k=1}^{n} \frac{k}{n}(\ln(k) - \ln(n)) = \frac{1}{n}(\sum_{k=1}^{n} k \ln k) - \frac{n+1}{2} \ln n.$$

This shows that $L = 1/2$ and so the correct answer is A. ■