Second Annual Columbus State Calculus Contest

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The Columbus State University
Department of Mathematics
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The Columbus State University Mathematics faculty welcome you to this year's Pre-Calculus/Calculus contest. We wish you success on this test and in your future studies.

Instructions

This is a 120-minute, 20-problem, multiple choice examination. There are five possible responses to each question. You should select the one "*best*" answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with a No. 2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response.

Throughout the exam, $\overline{AB}$ will denote the line segment from point A to point B and AB will denote the length of $\overline{AB}$. Pre-drawn geometric figures are not necessarily drawn to scale. The measure of the angle $\angle ABC$ is denoted by $m\angle ABC$.

The examination will be scored on the basis of $+23$ *for each correct answer,* -3 *for each incorrect selection, and* 0 *for each omitted item.*

No phones or any communication devices can be used. Calculators with CAS such as the TI-89 are not allowed. In fact, the test is designed in such a way that you do not really need a calculator. The problems denoted with $[\star^n]$ are tie-breaker problems, so more attention should be given to them. Possibly, include written justification for your answers, on the pages provided at the end of the test, especially for the tie-breaker problems. It is not necessary, but you may find useful reading the "Theoretical facts" part.

Do not open your test until instructed to do so!

Theoretical facts that you may find useful.

In this exam:

(1) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

(2) The inverse of a trigonometric function f may be indicated using the inverse function notation f^{-1} or with the prefix "arc" (e.g., $\sin^{-1} x = \arcsin x$).

(3) The derivative of a function f is denoted by f'. Second and third derivatives are denoted by f'' and f''' respectively.

Theorem 1: If a function, which is two times differentiable, satisfies $f''(x) \geq 0$ for x in some interval I, we can conclude that f is convex upward on I.

Theorem 2: A differentiable function for which $f'(x)$ changes sign at a, then f has a local extrema at a.

Theorem 3: The area of the region between the graphs of f and g (continuous functions), $x \in [a, b]$, is given by

$$A = \int_a^b |f(x) - g(x)|dx$$

Theorem 4: The fundamental theorem of calculus:

(a) If f is continuous on the interval $[a, b]$, then $F(x) = \int_a^x f(t)dt$ is differentiable and $F'(x) = f(x)$ for all $x \in [a, b]$.

(b) If f Riemann integrable on $[a, b]$ and F is an antiderivative of f, then $\int_a^b f(t)dt = F(b) - F(a)$.

Theorem 5: If f is differentiable and $f'(x) \geq 0$ on some interval, then f is non-decreasing on that interval.

Definition: A function is Riemann integrable on $[a, b]$ if the following limit exists and it is independent of the partition $\Delta = (x_0 = a, x_1, ..., x_{n-1}, x_n = b)$, and the points $c_i \in [x_{i-1}, x_i]$, $i = 1, 2, ..., n$:

$$\lim_{||\Delta|| \to 0} \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) = L, \ where \ ||\Delta|| = max\{x_i - x_{i-1} | i = 1, 2, ..., n\}.$$

If this is the case, L is written as $\int_a^b f(t)dt$.

Differentiation Rules: $(fg)' = f'g + fg'$, $(f/g)' = (f'g - fg')/g^2$, $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$ for all $|x| < 1$, $\frac{d}{dx} x^\alpha = \alpha x^{\alpha-1}$ for all $x > 0$, $(f \circ g)' = (f' \circ g)g'$, $(u^v)' = vu^{v-1}u' + u^v(\ln u)v'$, $\frac{d}{dx} \tan x = \sec^2 x$, $\frac{d}{dx} e^u = u'e^u$, $\frac{d}{dx}(\arctan)(x) = \frac{1}{1+x^2}$.

(MVT) Mean Value Theorem: If $F : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b), $F(b) - F(a) = (b - a)F'(c)$ for some $c \in (a, b)$.

Calculus Problems (Version Two)

1. Suppose that f is differentiable and defined on $\mathbb{R}$, $g(x) = \ln(x + \sqrt{x^2+1})$ for all real values of x, and $h = f \circ g$. If $f'(\ln 2) = 5$, what is $h'(\frac{3}{4})$?

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

2. The function $f(x) = \dfrac{x+3}{x^2+7}$ defined on the whole real line, has a maximum value of $f(a)$ and a minimum value of $f(b)$. Then, what is the value of $5a + b$?

   (A) -1 (B) -2 (C) -3 (D) -4 (E) -5

3. If for all real x we define $f(x) = e^{4x} - 8e^{2x} + 16x$, then the inverse of f, f^{-1}, exists and it is differentiable. Calculate $(f^{-1})'(-7) = \frac{d}{dx}(f^{-1})(-7)$.

   (A) 1 (B) 1/2 (C) 1/3 (D) 1/4 (E) 1/5

4. The equation of the tangent line to the graph of equation $(3x-2y)^2 + (x-y)^2 = 2$ at the point $(3, 4)$ is $y = mx + n$. What is $2m + n$?

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

5. The astroid

$$4(x^{\frac{2}{3}} + y^{\frac{2}{3}}) = 9$$

   has the property that each segment $\overline{AB}$ in the first quadrant, with A on the y-axis and B on the x-axis and tangent to the astroid, has the same length L. Find $\lfloor L \rfloor$ (the integer part of L) in this case.

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

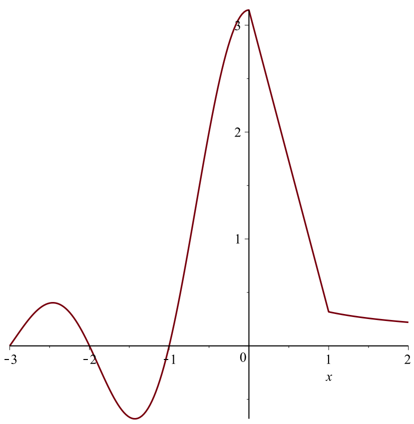
6. The function $F(x) = -x + x\ln(x)$ is defined for every $x > 0$. For every natural number n, The Mean Value Theorem applied to F on the interval $[a, b] = [e^n, e^{n+1}]$ gives $F(b) - F(a) = (b-a)F'(c)$ with $c \in (a, b)$. If $c = e^{n+\alpha}$, then what is $\frac{1}{\alpha}$?

   (A) $e-2$ (B) $e-1$ (C) e (D) $e+1$ (E) $e+2$

7. The values of a and b are chosen in such a way the function k, piecewisely defined by

$$k(x) = \begin{cases} \dfrac{\sin(2x)}{x} & \text{if } x < 0, \\ ax + b & \text{if } 0 \le x \le 1, \\ \dfrac{\ln(x)}{x-1} & \text{if } x > 1, \end{cases}$$

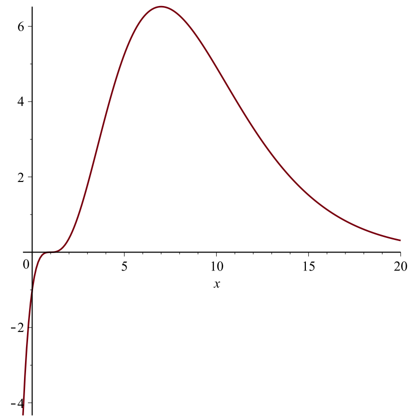
is continuous on the whole real line (graph as in the adjacent figure). What is $a + 2b$?



(A) 1 (B) 2 (C) 3

(D) 4 (E) 5

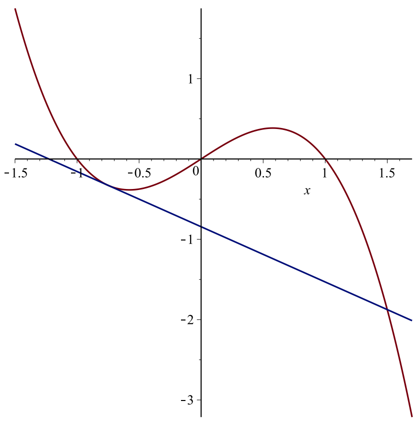
8. The function $f(x) = (x-1)^3 e^{-x/2}$ has inflection points at $x_1 = 1$, $x_2 = a + b\sqrt{3}$ and $x_3 = a - b\sqrt{3}$ with a and b natural numbers. Then what is $a - b$?



(A) 1 (B) 2 (C) 3

(D) 4 (E) 5

9. The cubic of equation $y = h(x) = x - x^3$ is shown in the figure on the right, together with its tangent line at the point $(-\frac{2}{5}, h(-\frac{2}{5}))$. This tangent line intersect the cubic at another point: $(\frac{m}{n}, h(\frac{m}{n}))$, where m and n are relatively prime natural numbers. Find $n - m$.



(A) 1 (B) 2 (C) 3

(D) 4 (E) 5

10. The function $g(x) = \frac{x^{2015}}{1-x}$ is defined for every $x \in (-1, 1)$. If $g^{(n)}$ denotes the n^{th} derivative of g, what is the value of $g^{(2014)}(0)$?

(A) 2014! (B) 2015! (C) -1 (D) 1 (E) 0

11. The polynomial function $f(x) = x^4 + mx^3 + 6x^2$, defined for all real values of x, has two distinct inflection points if and only if $|m| > a$. What is the value of a ?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

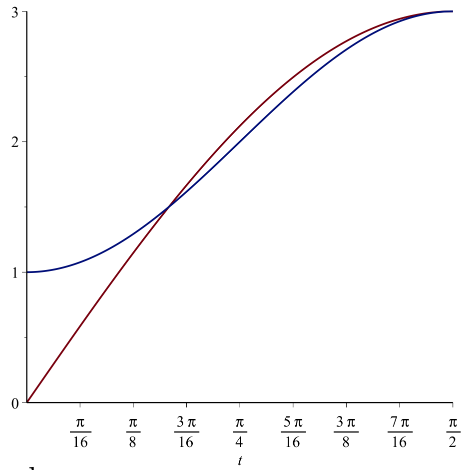
12. The graphs of $y = 3\sin x$ and $y = 2 - \cos(2x)$ for $x \in [0, \frac{\pi}{2}]$ are shown in the figure on the right. If A is the area between their graphs in this interval, then

$$A = \frac{m}{n}\sqrt{3} - \frac{2\pi}{3}$$

for a reduced fraction m/n. Calculate $m - n$.

(A) 1 (B) 2 (C) 3

(D) 4 (E) 5



13. For m and n relatively prime positive integers, we have

$$\lim_{x \to \pi} |\sin x + \cos x|^{\sec(\frac{x}{2})} = e^{-m/n}.$$

What is $m + 3n$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

14. Consider the function $F(x) = \displaystyle\int_{\ln x}^{\ln(x+1)} \frac{1}{1+e^{2t}} dt$, for x in $(0, 2)$. Find $-5F'(1)$.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

15. We define f by the rule $f(x) = (\sin x)^4 + (\cos x)^4$ for all real numbers x. Knowing that c is the smallest positive number with the property

$$f(c) = \frac{1}{2\pi} \int_0^{2\pi} f(x)dx$$

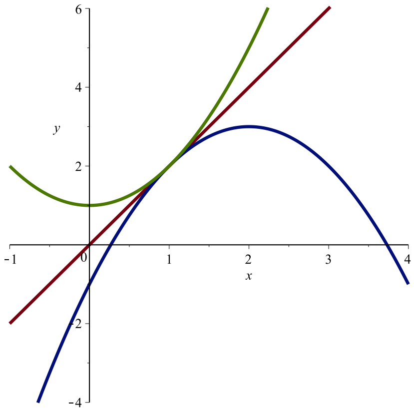
find $\frac{\pi}{c}$.

(A) 12 (B) 10 (C) 8 (D) 6 (E) 4

16. The parabolas $y = x^2 + 1$ and $y = 19 - (x - a)^2$ are tangent to one another and tangent to the line $y = mx + n$ (as in the Figure on the right). Knowing that $a > 0$, find $2m + n$.

(A) 1 (B) 2 (C) 3

(D) 4 (E) 5



17. [$*^1$] What is the smallest value of $x^2 + y^2$ if $x^2 + 3xy + 5y^2 = 11$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

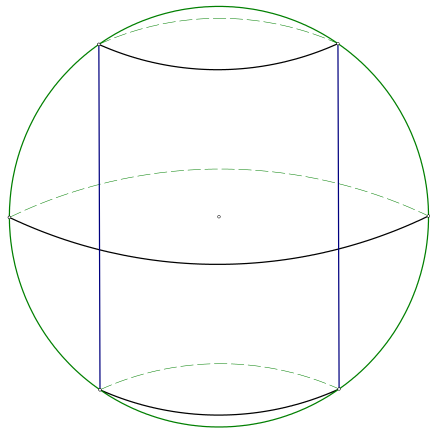
18. [$*^2$] The function E satisfies the differential equation $E(t)^3 E''(t) = -1$ for every $t \leq 0$ and the initial conditions $E(0) = 1$, $E'(0) = -1$. What is the value of $E(-4)$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

19. [$*^3$] A cylinder is inscribed in a sphere of radius 3 inches as in the adjacent figure. What is the maximum volume of such a cylinder (in cubic inches)?

(A) $4\pi\sqrt{3}$ (B) $6\pi\sqrt{3}$ (C) $8\pi\sqrt{3}$

(D) $10\pi\sqrt{3}$ (E) $12\pi\sqrt{3}$



20. [$*^4$] We denote by L the following limit:

$$L = \lim_{n\to\infty}\left(\frac{n+1}{n}\ln n - \frac{2}{n^2}\sum_{k=1}^{n} k\ln k\right).$$

Find $L/(1-L)$.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5