Third Annual Columbus State Calculus Contest-Precalculus Test

Sponsored by
The Columbus State University
Department of Mathematics
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The Columbus State University Mathematics faculty welcome you to this year's Pre-Calculus/Calculus contest. We wish you success on this test and in your future studies.

Instructions

This is a 120-minute, 10-problem, multiple choice examination. There are five possible responses to each question. You should select the one "*best*" answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with a No. 2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response.

Throughout the exam, $\overline{AB}$ will denote the line segment from point A to point B and AB will denote the length of $\overline{AB}$. Pre-drawn geometric figures are not necessarily drawn to scale. The measure of the angle $\angle ABC$ is denoted by $m\angle ABC$.

The examination will be scored on the basis of $+23$ *for each correct answer,* -3 *for each incorrect selection, and* 0 *for each omitted item.*

No phones or any communication devices can be used. Calculators with CAS such as the TI-89 are not allowed. In fact, the test is designed in such a way that you do not really need a calculator. The problems denoted with $[\star^n]$ are tie-breaker problems, so more attention should be given to them. Possibly, include written justification for your answers, on the pages provided at the end of the test, especially for the tie-breaker problems. It is not necessary, but you may find useful reading the "Theoretical facts" part.

Do not open your test until instructed to do so!

Theoretical facts that you may find useful.

Theorem 1: (**Factor Theorem**) Given a polynomial P, then P is divisible by $x-a$ if and only if $P(a)=0$. The remainder of the division of $P(x)$ by $x-a$ is $P(a)$.

Theorem 2: The trigonometric addition/subtraction formulae:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta, \quad \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Double Angle Formulae: $\sin 2\alpha = 2\sin\alpha\cos\alpha$, $\cos 2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$.

Theorem 3: A polynomial $P(x) = a_0 + a_1x + \cdots a_nx^n$ has a rational zero, p/q (reduced form), if p divides a_n and q divides a_0.

Theorem 4: Quadratic formula and Viete's relations: the equation $ax^2+bx+c=0$ has two zeros (x_1, x_2, real or complex) given by

$$\boxed{x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}}.$$

These zeros satisfy Viete's relations:

$$\boxed{x_1 + x_2 = -\frac{b}{a}, \quad x_1x_2 = \frac{c}{a}}.$$

Theorem 5: Heron's formula: The area of a triangle is given by the formulae

$$A = \sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{4}\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}$$

where $s = \frac{a+b+c}{2}$ and a, b and c are the sides of the triangle.

Theorem 6: The Law of Sines in a triangle states that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin A}$$

where A, B and C are the measurement of the interior angles of the triangle.

Theorem 7: (**Pythagorean Theorem:**) In a right triangle with legs b, c and hypothenuse a, we have $b^2+c^2=a^2$.

Theorem 8: Properties of logarithms:

$$\log_a u = v \Leftrightarrow u = a^v$$

$$\log_a u + \log_a v = \log_a uv, \quad \log_a u^v = v\log_a u$$

Pre-calculus Problems (Version I)

1. Given two positive real numbers x and y such that $\ln(x^2y^3) = 1$ and $\ln(x^5y^7) = 1$, find $\ln(xy^3)$.

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

2. The equation $3^{3x-3} + 3^{2x-2} + 3^{x-2} = 1$ has only one real solution, say x_0. The solution can be written $x_0 = \log_3[(10m+n)^{1/3} - 1]$ for some natural numbers m and n less than 10. What is $n - 3m$?

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

3. If a and b are real numbers such that $a > b > 0$, and $\dfrac{a^3 - b^3}{(a-b)^3} = 4$, then the ratio $\frac{a}{b}$ can be written as $\frac{a}{b} = \frac{m+\sqrt{n}}{2}$ for some natural numbers m and n. Find $n - m$.

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

4. The remainder of the division $\left(\frac{5-2x^2}{3}\right)^{2015} \div (x^2 - x - 2)$ is of the form $A + Bx$. Find $A - 4B$.

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

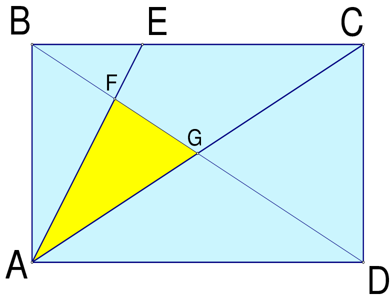
5. The quartic equation $x^4 + 2x^3 - 10x^2 - 2x + 1 = 0$ has four real solutions of the form $x_1 = 1 + \sqrt{n}$, $x_2 = 1 - \sqrt{n}$, $x_3 = p + \sqrt{m}$ and $x_3 = p - \sqrt{m}$ for some integers m, n and p. Find $m - 2n$.

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

6. For x in the interval $(6, 11)$ the equation in t, $|t+4| - |t-2| + |t-7| = x$, has four solutions written in increasing order, $t_1 < t_2 < t_3 < t_4$. The expression $t_3t_4 - t_1t_2$ can be simplified to $mx + n$ for some whole numbers m and n. Find the value of m.

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

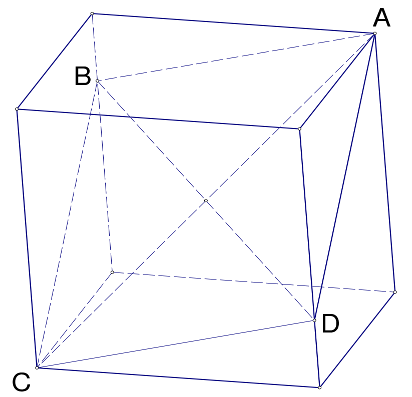
7. In the accompanying figure we have a rectangle $ABCD$ and G the intersection of its diagonals. We know that $\overline{AE}$ and $\overline{AC}$ are trisecting the angle $\angle BAD$. Let F be the intersection of the diagonal $\overline{BD}$ and $\overline{AE}$. Knowing that $AB = 1$ then what is the ratio $\frac{Area(ABCD)}{2Area(AFG)}$?



(A) 1 (B) 2 (C) 3

(D) 4 (E) 5

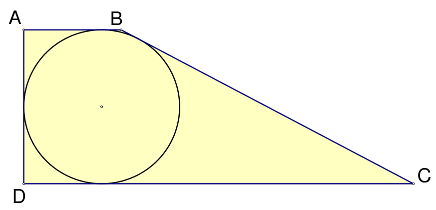
8. [∗[1]] In the accompanying figure we have a section ABCD into a cube of side-lengths 1, which cuts the cube along the diagonal $\overline{AC}$, and points B and D divide the respective sides into ratios (top to bottom) $1 : 2$ and $2 : 1$. What is the area of $ABCD$?



(A) $\frac{3\sqrt{13}}{8}$ (B) $\frac{\sqrt{14}}{4}$ (C) $\frac{\sqrt{14}}{3}$

(D) $\frac{\sqrt{15}}{3}$ (E) $\frac{\sqrt{13}}{3}$

9. [∗[2]] In the trapezoid ABCD with bases $\overline{AB}$ and $\overline{DC}$, two of its side lengths are $AD = 8$ and $BC = 17$. Knowing that $m(\angle ADC) = 90°$ and the trapezoid $ABCD$ is cyclic (there exists a circle tangent to all sides-see figure on the right), find the area of ABCD.



(A) 60 (B) 70 (C) 80

(D) 90 (E) 100

10. [∗[3]] The trigonometric equation $8\cos(x)\cos(2x)\cos(4x) = 1$, has exactly three solutions in the interval $(0, \frac{\pi}{2})$ (measured in radians). If we write these three solutions in increasing order, $x_1 < x_2 < x_3$, then the expression $\frac{x_1+x_2-x_3}{\pi}$ is a positive rational number which in reduced form, $\frac{a}{b}$, leads to what number at the numerator?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5