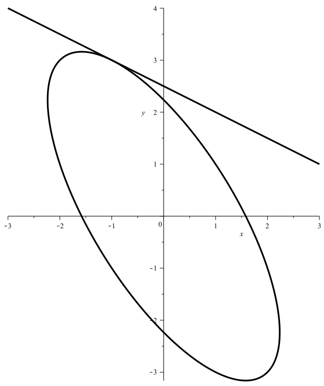
Solutions to the Fourth Annual Columbus State Calculus Contest

April 29^{th}, 2016

1. *The equation of the tangent line to the graph of equation*

$$(3x+2y)^2+(x-y)^2=25$$

at the point $(-1,3)$ *is* $y = mx + n$. *What is* $\frac{n}{m}+7$?



(A) 1 (B) 2 (C) 3

(D) 4 (E) 5

Solution: We use implicit differentiation to get $2(3x+2y)(3+2y')+2(x-y)(1-y')=0$. Substituting $x=-1$ and $y=3$ gives $3(3+2y')-4(1-y')=0$. Solving for y', we obtain $y'=-1/2$. Hence the equation of the tangent line is $y=3-(x+1)/2$ or $\boxed{y=-\frac{1}{2}x+\frac{5}{2}}$. Hence, the answer is $\boxed{B}$. ■

2. *The function* $F(x)=\frac{1}{x}$ *is defined for every* $x>0$. *The Mean Value Theorem applied to* F *on the interval* $[a,b]$ *gives* $F(b)-F(a)=(b-a)F'(c)$ *with* $c\in(a,b)$. *For* $a=2$ *and* $b=8$, *then what is* c?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: Since $F'(x)=-\frac{1}{x^2}$, we have $\frac{1}{b}-\frac{1}{a}=-(b-a)\frac{1}{c^2}$. After simplifying by $b-a$ we get $c^2=ab$ or $c=\sqrt{ab}$. In our case $c=\sqrt{2(8)}=4$. Therefore, we see that the correct answer is $\boxed{D}$. ■

3. *The function* $g(x)=\frac{1}{x(1-x)}$ *is defined for every* $x\in(0,1)$. *If* $g^{(n)}$ *denotes the* n^{th} *derivative of* g, *and*

$$g^{(2016)}\left(\frac{1}{2}\right)=(2016!)2^{2016+\alpha}$$

what is α?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: We observe that $g(x)=\frac{1}{x}+\frac{1}{1-x}=x^{-1}-(x-1)^{-1}$. We notice that $g'(x)=-x^{-2}+(x-1)^{-2}$, $g''(x)=2![x^{-3}-(x-1)^{-3}]$, and by induction we see that

$g^{(n)}(x) = (-1)^n n![x^{-(n+1)} - (x-1)^{-(n+1)}]$ for every n natural number. In particular, $g^{(2016)}\left(\frac{1}{2}\right) = (2016!)(2^{2017} + 2^{2017}) = (2016!)2^{2018}$ which implies that $\alpha = 2$. This gives $\boxed{B}$ as the correct answer. ■

4. *Suppose that f is defined on $\mathbb{R}$ by the rule*

$$f(x) = x\sqrt{x^2+1} + \ln(x + \sqrt{x^2+1}).$$

If $h = f \circ f$, what is $h'(0)$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: Using the chain rule, we get $h' = (f' \circ f)f'$. So we need do compute

$$f'(x) = \sqrt{x^2+1} + \frac{x^2}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}} = 2\sqrt{x^2+1}.$$

Hence, $h'(0) = f'(f(0))f'(0) = f'(0)^2 = 2^2 = 4$. Ergo, the answer is $\boxed{D}$. ■

5. *The function $G(x) = (x^2+m)e^x$, defined for all real values of x and m a fixed real valued constant, has two distinct inflection points if and only if $m < a$. What is the value of a ?*

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: We calculate the second derivative and obtain $G''(x) = (x^2+m+4x+2)e^x$. In order to have two distinct inflection points, we need have two distinct real solutions of the quadratic $x^2+4x+m+2 = 0$. It is then equivalent to requiring that its determinant $\Delta = 16 - 4(m+2) = 8 - 4m > 0$ or $m < 2$. So, the answer is $\boxed{B}$. ■

6. *The function $g(x) = \dfrac{x+1}{x^2+8}$ defined on the whole real line, has a maximum value of $g(a)$ and a minimum value of $g(b)$. Then, what is the value of $(a-b)/2$?*

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: We calculate the derivative, $g'(x) = -\frac{(x+4)(x-2)}{(x^2+7)^2}$, and observe that $a = 2$ and $b = -4$. Hence $(a-b)/2 = 3$ and the correct answer is $\boxed{C}$. ■

7. *For m a positive integer, we have*

$$\lim_{x\to 0} |\cos x + \sin 2x + \sin 3x|^{\cot x} = e^m.$$

What is m?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: The limit is equivalent to $\lim_{x\to 0} \cot x \ln|\cos x + \sin 2x + \sin 3x| = m$. Using L'Hospital's Rule, we have

$$m = \lim_{x\to 0} \frac{\sin x + 2\cos 2x + 3\cos 3x}{\cos x + \sin 2x + \sin 3x} = 5$$

This implies the answer is $\boxed{E}$. ■

8. *If for all real x we define $f(x) = 6x^5 - 15x^4 + 10x^3$, then the inverse of f, f^{-1}, exists and it is differentiable. The derivative of the inverse function at $1/2$ is a rational number, i.e.,*

$$(f^{-1})'(\frac{1}{2}) = \frac{d}{dx}(f^{-1})(\frac{1}{2}) = \frac{m}{n},$$

which is written in reduced form using two natural numbers $m, n \in \mathbb{N}$. What is $2m-n$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: Let us calculate the derivative of f:

$$f'(x) = 30x^4 - 60x^3 + 30x^2 = 30x^2(x-1)^2 \geq 0$$

and so f is strictly increasing. Since $(f^{-1} \circ f)(x) = x$, we can differentiate this equality with respect to x and substitute $x = 1/2$ in what we get: $(f^{-1})'(f(1/2))f'(1/2) = 1$. This implies $(f^{-1})'(1/2) = \frac{8}{15}$, so the answer is $\boxed{A}$. ■

9. *The number k is the smallest positive number with the property*

$$|\sin x - 2\cos x|^2 \leq k$$

for all real numbers x. Find the value of k.

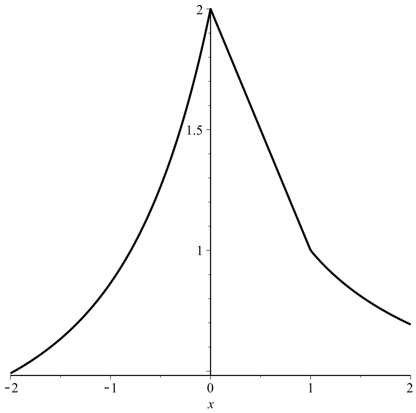
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: If we denote by α the angle in the first quadrant, such that $\sin\alpha = \frac{2}{\sqrt{5}}$. It follows that $\cos\alpha = \frac{1}{\sqrt{5}}$. with this notation we can change the inequality to

$$(\sin x \cos\alpha - \sin\alpha \cos x)^2 \leq \frac{k}{5} \Leftrightarrow \sin(x-\alpha)^2 \leq \frac{k}{5}.$$

This shows that $k = 5$, which makes $\boxed{E}$ the correct answer. ■

10. *The values of a and b are chosen in such a way the function k, piecewisely defined by*



$$k(x) = \begin{cases} \dfrac{e^{2x}-1}{x} & \text{if } x < 0, \\ ax + b & \text{if } 0 \leq x \leq 1, \\ \dfrac{\ln x}{x-1} & \text{if } x > 1, \end{cases}$$

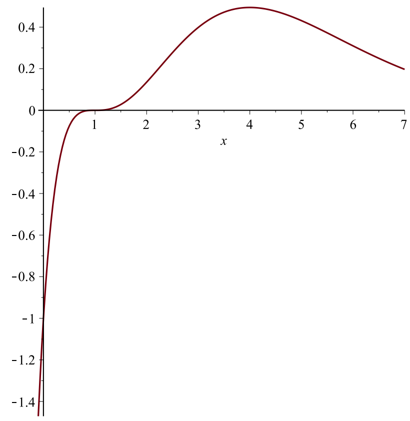
is continuous on the whole real line (graph as in the adjacent figure). What is $a + 2b$?

(A) 1 (B) 2 (C) 3

(D) 4 (E) 5

Solution: Using L'Hospital's Rule we get that $\lim_{x \nearrow 0} k(x) = \lim_{x\to 0} 2e^{2x} = 2$ which means that $b = 2$. Similarly, $\lim_{x \searrow 1} k(x) = \lim_{x\to 1}(1/x) = 1$, which means $a + b = 1$ and so $a = -1$. Therefore the answer is $\boxed{C}$. ■

11. *The function $f(x) = (x-1)^3 e^{-x}$ has inflection points at $x_1 = 1$, $x_2 = a + b\sqrt{3}$ and $x_3 = a - b\sqrt{3}$ with a and b natural numbers. Then what is $a - 3b$?*

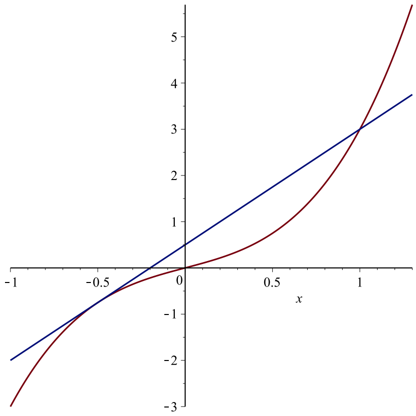


(A) 1 (B) 2 (C) 3

(D) 4 (E) 5

Solution: First, we calculate the derivative $f'(x) = (4-x)(x-1)^2 e^{-x}$ and then the second derivative $f''(x) = e^{-x}(x^2 - 8x + 13)(x-1)$. Solving for $f''(x) = 0$ gives $x_1 = 1$ and $x_{2,3} = 4 \pm \sqrt{3}$. Hence, the answer is $\boxed{A}$. ■

12. *The* cubic *of equation* $y = h(x) = x + 2x^3$ *is shown in the figure on the right, together with its tangent line at the point* $(-\frac{3}{4}, h(-\frac{3}{4}))$. *This tangent line intersects the cubic at another point:* $(\frac{m}{n}, h(\frac{m}{n}))$, *where* m *and* n *are relatively prime natural numbers. Find* $2m - n$.



(A) 1 (B) 2 (C) 3

(D) 4 (E) 5

Solution: Assuming that $h(x) = c_0x^3 + c_1x^2 + c_2x + c_3$, using the Taylor polynomial to approximate h we get that this is in fact an exact formula:

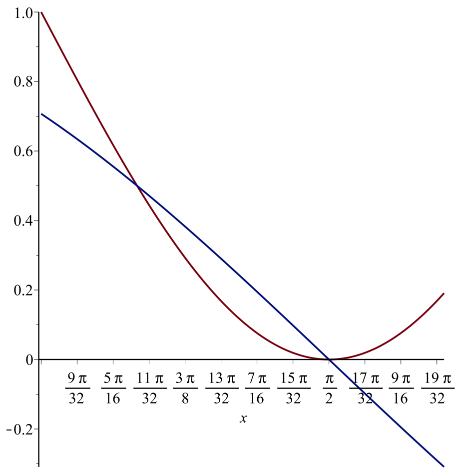
$$h(x) = h(a) + h'(a)(x-a) + \frac{1}{2}h''(a)(x-a)^2 + \frac{1}{6}h'''(a)(x-a)^3,$$

for any point a. So, if we want to solve the equation $h(x) = h(a) + h'(a)(x-a)$, we obtain equivalently, $x = a$ a double root, and the equation $3c_0a + c_1 + c_0(x-a) = 0$ or $x = -2a - \frac{c_1}{c_0}$. So, in our particular case the tangent line intersects the curve $y = h(x)$ again at $(\frac{3}{2}, h(\frac{3}{2}))$. Thus, $2m - n = 4$ and so the correct answer is $\boxed{D}$. ■

13. The graphs of $y = \cos x$ and $y = 1 + \cos(2x)$ for $x \in [\frac{\pi}{4}, \frac{3\pi}{5}]$ are shown in the figure on the right. If A is the area between their graphs in this interval, then

$$A = 1 - \frac{m}{n}\sqrt{3} - \frac{\pi}{6}$$

for a reduced fraction m/n, where m and n are natural numbers. What is n?



(A) 1 (B) 2 (C) 3

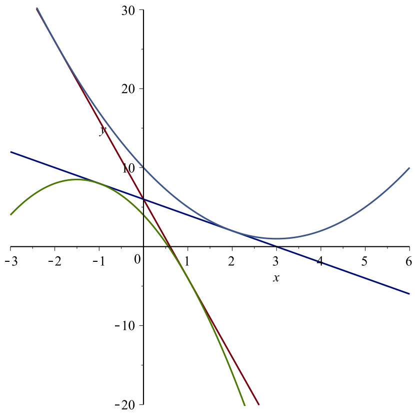
(D) 4 (E) 5

Solution: We need to see where the two graphs intersect: $\cos x = 1 + \cos 2x$ or using one of the double angle formulas, this is equivalent to $\cos x = 2\cos^2 x$. So, we get $\cos x = 0$ or $\cos x = 1/2$. Thus, we need to integrate between $\pi/3$ and $\pi/2$. Hence, $A = \int_{\pi/3}^{\pi/2} (\cos x - 1 - \cos 2x)dx$ which implies

$$A = \sin x|_{\pi/3}^{\pi/2} - x|_{\pi/3}^{\pi/2} - \frac{\sin 2x}{2}|_{\pi/3}^{\pi/2} = 1 - \frac{1}{4}\sqrt{3} - \frac{\pi}{6}.$$

This shows that $\boxed{D}$ is the correct answer. ■

14. [*[1]] *The parabolas $y = x^2 - 6x + 10$ and $y = 4 - 6x - 2x^2$ have two lines that are tangent to both of them (as in the accompanying figure). If the equations of these tangent lines are $y = mx + n$ and $y = px + q$ with $k = m/p > 1$, what is k?*



(A) 1 (B) 2 (C) 3

(D) 4 (E) 5

Solution: Assuming that the line of equation $y = mx+n$ is tangent to both parabolas, it means that the polynomials $x^2 - 6x + 10 - mx - n = x^2 - (m+6)x + 10 - n$ and $2x^2 + 6x - 4 + mx + n = 2x^2 + (6+m)x + n - 4$ are both perfect squares. This is equivalent to $(m+6)^2 - 4(10-n) = (6+m)^2 - 4(2)(n-4) = 0$. This system gives $n = 6$ and $m_1 = -2$ or $m_2 = -10$. Hence the answer is $\boxed{E}$. ■

15. *The nonzero function E satisfies the differential equation $E'(t) = E(t)^{2016}$ for every t real number and the initial conditions $E(0) = 1$. If*

$$E(-1)^{-2015} = 2015 + \alpha$$

what is the value of α?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: The equation is equivalent to $E'(t)E(t)^{-2016} = 1$, which can be integrated to $\frac{1}{2015}(1 - E(t)^{-2015}) = t$. If we plug in $t = -1$ we obtain $\alpha = 1$. Hence, the answer is $\boxed{A}$. ■

16. *Consider the function $F(x) = \displaystyle\int_{\tan x}^{\tan 2x} \frac{1}{(1+t^2)^2} dt$, for x in $(0, \pi/4)$. If for every x we have*

$$F'(x) = a(\cos x)^4 + b(\cos x)^2 + c$$

for some constants a, b and c, what is $a + b + c$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: Using the Fundamental Theorem of Calculus we get

$$F'(x) = \frac{1}{(1+\tan^2 2x)^2} 2\sec^2 2x - \frac{1}{(1+\tan^2 x)^2}\sec^2 x,$$

or,

$$F'(x) = 2\cos^2 2x - \cos^2 x = 2(2\cos^2 x - 1)^2 - \cos^2 x = 8\cos^4 x - 9\cos^2 x + 2.$$

So, $\boxed{A}$ is the correct answer. ■

17. *The recurrent sequence $\{x_n\}$ satisfies the recurrence $x_{n+1} = 2x_n(1-x_n)$ for every $n \geq 1$ and $x_1 = 1/2016$. Knowing that $\{x_n\}$ is convergent to L, what is L^{-1}?*

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: Since x_{n+1} converges to L, we have the equation in L: $L = 2L(1-L)$. We either get $L = 0$ or $L = \frac{1}{2}$. It is easy to see that x_n is non-decreasing. Indeed, $x_{n+1} \leq [x_n + (1-x_n)]^2/2 = 1/2$ and so $x_{n+1} \geq x_n$ is equivalent to $2(1-x_n) \geq 1$ or $x_n \leq 1/2$. This implies $L^{-1} = 2$ and then the answer is $\boxed{B}$. ■

18. *We define f by the rule $f(x) = (\sin x)^6 + (\cos x)^6$ for all real numbers x. Knowing that c is the smallest positive number with the property*

$$f(c) = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$$

find $\frac{\pi}{c}$.

(A) 12 (B) 10 (C) 8 (D) 6 (E) 4

Solution: We can simplify f in the following way

$$f(x) = (\sin^2 x)^3 + (\cos^2 x)^3 = \sin^4 x - \sin^2 x\cos^2 x + \cos^4 x = 1 - 3\sin^2 x\cos^2 x, \Leftrightarrow$$

$$f(x) = 1 - \frac{3}{4}\sin^2 2x = 1 - \frac{3}{8}(1-\cos 4x) = \frac{5}{8} + \frac{3}{8}\cos 4x.$$

So, $\int_0^{2\pi} f(x)dx = \frac{5}{8}(2\pi)$. Then the given equation in c is equivalent to $\cos(4c) = 0$. The smallest positive solution of this equation is clearly given by $4c = \pi/2$, which attracts $\pi/c = 8$. Hence, $\boxed{C}$ is the answer. ■

19. [$*^2$] *We denote by L the following limit:*

$$L = \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2 e^{\frac{k}{n}}.$$

Find $e - L$.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: We use the Riemann Sums definition of the definite integral for $f(x) = x^2e^x$ on the interval $(0, 1]$. We can compute easily $\int_0^2 f(x)dx = e^x(x^2 - 2x + 2)|_0^1 = e - 2$. Hence

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k^2}{n^2} e^{\frac{k}{n}} = e - 2.$$

This shows that $L = e - 2$ and so the correct answer is $\boxed{B}$. ■

20. [$*^3$] *What is the smallest value of* $2x^2 + y^2$ *if* $x^2 + 6xy + 4y^2 = 25$*?*

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: We think of a point $(x, y(x))$ on the curve $x^2 + 6xy + 4y^2 = 25$, and as a function of x we need to minimize $2x^2 + y(x)^2$. Hence we look for critical points, or $4x + 2yy' = 0$. This implies $y' = -2x/y$. Using implicit differentiation we get $2x + 6y + 6xy' + 8yy' = 0$ or $6x^2 + 7xy - 3y^2 = 0$. This homogeneous equation can be solved by factorization $(2x + 3y)(3x - y) = 0$. Then $y = 3x$ or $y = -2x/3$.

First, we look at $y = 3x$: $x^2 + 6x(3x) + 4(9x^2) = 25$ or $x^2 = \frac{5}{11}$. This gives $2x^2 + y^2 = \frac{10}{11} + \frac{9(5)}{11} = 5$.

The other case gives no intersection with the curve. The curve is a hyperbola ($(x + 3y)^2 - 5y^2 = 25$) and so $2x^2 + y^2$ can be as large as we want. Therefore, the minimum is 5 and $\boxed{E}$ is the correct answer. ■