Solutions to the Fourth Annual Columbus State Calculus Pre-calculus contest

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Department of Mathematics
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1. *Given two positive real numbers x and y such that $x^2y^5 = 256$ and $x^3y^8 = 8192$, find xy.*

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: The first equality is equivalent to $x^2y^5 = 2^8$ or after raising everything to power 5 we get $x^{10}y^{25} = 2^{40}$. The second equality is the same as $x^3y^8 = 2^{13}$ or after exponentiating to power 3 we obtain $x^9y^{24} = 2^{39}$. Diving these equalities gives

$$xy = \frac{x^{10}y^{25}}{x^9y^{24}} = \frac{2^{40}}{2^{39}} = 2.$$

So, the answer is $\boxed{B}$. ■

2. *What is the product of all real x such that $(4^x - 8)^3 + (8^x - 4)^3 = (4^x + 8^x - 12)^3$?*

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: If we set $a = 4^x - 8$ and $b = 8^x - 4$, we observe that the given equation is equivalent to $a^3 + b^3 = (a + b)^3$. This simplifies to $a^3 + b^3 = a^3 + 3a^2b + 3ab^2 + b^3$ or $3ab(a + b) = 0$. Hence, the only solutions are $a = 0$, $b = 0$ or $a = -b$. The first implies $x = 3/2$, the second implies $x = 2/3$. The last equality can be written as $2^{3x} + 2^{2x} - 12 = 0$ or in factor form (we see that $x = 1$ is a particular solution), $(2^x - 2)[2^{2x} + 3(2^x) + 6] = 0$. Since $2^x > 0$ we observe that $x = 1$ is the only solution. So, $\boxed{A}$ is the correct answer since $(3/2)(2/3)(1) = 1$. ■

3. [*[1]] *The trigonometric equation*

$$\frac{\sqrt{3}-1}{\sin x} + \frac{\sqrt{3}+1}{\cos x} = 4\sqrt{2},$$

has two solutions in the interval $(0, \frac{\pi}{2})$, say α and β (written in radians) with $\alpha < \beta$. We have $\frac{\beta}{\alpha} = \frac{m}{n}$, for m and n relatively prime positive integers. What is $m - 3n$?

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: If we divide by 2 and observe that $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\cos\frac{\pi}{3} = \frac{1}{2}$, the equation can be written as

$$\frac{\cos\frac{\pi}{6} - \cos\frac{\pi}{3}}{\sin x} + \frac{\cos\frac{\pi}{6} + \cos\frac{\pi}{3}}{\cos x} = 4\sin\frac{\pi}{4}.$$

Now using the formulae $\cos a - \cos b = 2\sin\frac{b-a}{2}\sin\frac{a+b}{2}$ and $\cos a + \cos b = 2\cos\frac{a-b}{2}\cos\frac{a+b}{2}$, we can continue the above as

$$\frac{2\sin\frac{\pi}{12}\sin\frac{\pi}{4}}{\sin x} + \frac{2\cos\frac{\pi}{12}\cos\frac{\pi}{4}}{\cos x} = 4\sin\frac{\pi}{4}.$$

Since $\sin\frac{\pi}{4} = \cos\frac{\pi}{4}$ we can simplify by $2\cos\frac{\pi}{4}$ and after eliminating the denominators we obtain

$$\sin\frac{\pi}{12}\cos x + \cos\frac{\pi}{12}\sin x = 2\sin x\cos x \Leftrightarrow \sin(\frac{\pi}{12} + x) = \sin 2x.$$

From here we can solve for all the solutions, $\frac{\pi}{12} + x = 2x$ gives $\alpha = \frac{\pi}{12}$, and $\frac{\pi}{12} + x = \pi - 2x$ gives $\beta = \frac{11\pi}{36}$. Therefore, $\frac{\beta}{\alpha} = \frac{11}{3}$ which shows that $m - 3n = 2$, so the answer is $\boxed{B}$. ■

4. *The positive integer M has two digits, i.e., $M = 10a + b$ for some a and b in $\{0, 1, ..., 9\}$. Knowing that*

$$1 + 2 + ... + M = 2016$$

what is $a - b$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: We need to solve the quadratic equation $M(M+1)/2 = 2016$ or $M^2 + M = 4032$. Multiplying by 4 and adding both sides 1, turns this into $(2M+1)^2 = 16129 = 127^2$. Hence, $M = (127-1)/2 = 63$ (the second solution is a negative integer). Then $a = 6$ and $b = 3$ which shows that $\boxed{C}$ is the correct choice here. ■

5. *Find the number of ordered pairs of integers (a, b), $1 < a \leq 15$, such that*

$$a^2 + b^2 - ab^2 - ab + b = 1.$$

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: The equation can be written as $(a-1)(a+1) - b^2(a-1) - b(a-1) = 0$ and since we have $a > 1$, we can simplify it to $a + 1 = b^2 + b$. As in the Problem 4, we can write this as $4a + 5 = (2b+1)^2$. So, $4a + 5$ is a perfect odd square between 13 and 65: 5^2 and 7^2. This gives the pairs $(5, 2)$, $(5, -3)$, $(11, 3)$, and $(11, -4)$. Therefore, the answer is $\boxed{D}$. ■

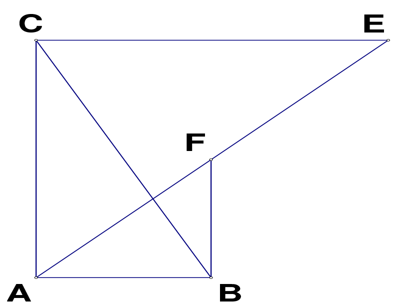
6. *Find the number of solutions (x, y) (ordered pairs of real numbers) of the system*

$$\begin{cases} |x+y+1| = |x-y+2| \\ |x-2y+3| = |2x-y+4| \end{cases} .$$

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: The first equation is equivalent to $x+y+1 = \pm(x-y+2)$. So, we get either $y = \frac{1}{2}$ or $x = -\frac{3}{2}$. If $y = \frac{1}{2}$, the second equation is equivalent to $x+2 = \pm(2x+\frac{7}{2})$. This leads to either $x = -\frac{3}{2}$ or $x = -\frac{11}{6}$. So, in this case the solutions are $(-\frac{3}{2}, \frac{1}{2})$ and $(-\frac{11}{6}, \frac{1}{2})$.
If $x = -\frac{3}{2}$, the second equation becomes $\frac{3}{2} - 2y = \pm(1-y)$. Form here, we either get $y = \frac{1}{2}$ which leads to the same first pair as before, or $y = \frac{5}{6}$. Hence, the number of solutions is 3, and the correct answer is $\boxed{C}$. ■

7. [$*^2$] *In the accompanying figure we have a right triangle $\triangle ABC$, $\angle A = 90°$, $m(\angle B) > m(\angle C)$ (measured in degrees). Perpendiculars on $\overline{AB}$ and $\overline{AC}$ at B and C respectively, intersect the angle bisector of the angle $\angle A$ at F and E respectively. Knowing that $EF = BC$, what is $\frac{m(\angle B)}{m(\angle C)}$?*



(A) 1 (B) 2 (C) 3

(D) 4 (E) 5

Solution: The two triangles $\triangle ABF$ and $\triangle ACE$ are isosceles right triangles. So, if the sides of $\triangle ABC$ are denoted as usual $AB = c$, $AC = b$ and $BC = a$ then the information given translates into

$$b^2 + c^2 = a^2 = (AE - AF)^2 = (b\sqrt{2} - c\sqrt{2})^2 = 2(b^2 - 2bc + c^2) \Leftrightarrow$$

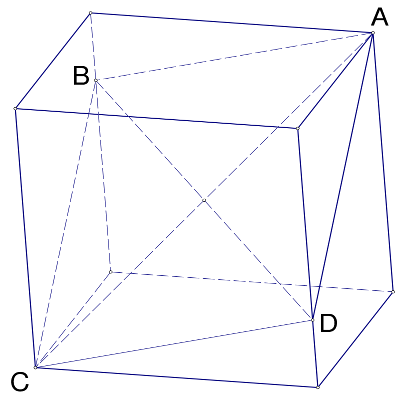
$$b^2 - 4bc + c^2 = 0 \Leftrightarrow \frac{c}{b} = 2 \pm \sqrt{3}.$$

By hypothesis $\frac{c}{b} < 1$, so $\tan \angle C = \frac{c}{b} = 2 - \sqrt{3}$. This implies

$$\tan 2\angle C = \frac{2\tan \angle C}{1 - \tan^2 \angle C} = \frac{\sqrt{3}}{3} = \tan 30°.$$

Hence, $m(\angle C) = 15°$ and $m(\angle B) = 75°$. Since $m(\angle B) = 5m(\angle C)$ we see that the correct answer is $\boxed{E}$. ■

8. [$*^3$] In the accompanying figure we have a section ABCD into a cube of side-lengths 1, which cuts the cube along the diagonal $\overline{AC}$, and points B and D divide the respective sides into ratios (top to bottom) $3:5$ and $5:3$. The area of $ABCD$ is equal to $\frac{m\sqrt{2}}{n}$ with m and n relatively prime positive integers. What is $n-m$?



(A) 1 (B) 2 (C) 3

(D) 4 (E) 5

Solution: Let us work with the general situation, i.e., the ratio given is simply a number t. One can use Heron's Formula to compute the area of the triangle ACD since $AC=\sqrt{3}$, $AD=BC=\sqrt{1+t^2/(1+t)^2}$ and $CD=\sqrt{1+1/(1+t)^2}$. In this case it is useful to use a different version of Heron's Formula:

$$A=\frac{1}{4}\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)},\ or$$

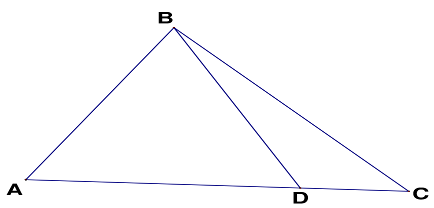
$$A=\frac{1}{4}\sqrt{2a^2(b^2+c^2)-a^4-(b^2-c^2)^2}.$$

We observe that we can set $a^2=3$, $b^2=1+t^2/(1+t)^2$ and $c^2=1+1/(1+t)^2$ in the above formula. This gives $A^2=\frac{1}{16}[\frac{8(t^2+t+1)}{(t+1)^2}]$. Hence, we obtain the general formula

$$Area(ABCD)=2A=\frac{\sqrt{2(t^2+t+1)}}{t+1}.$$

In this case $t=5/3$ and so because $5^2+5(3)+3^2=49=7^2$ we get $Area(ABCD)=\frac{7}{8}\sqrt{2}$. Therefore, the correct answer is $\boxed{A}$. ■

9. [$*^4$] In the triangle $\triangle ABC$, $AB=5$, $BC=7$, $AC=9$ and D is on $\overline{AC}$ with $BD=5$. The ratio $\frac{AD}{DC}=\frac{m}{n}$ with m and n relatively prime positive integers. What is $3n-m$?



(A) 1 (B) 2 (C) 3

(D) 4 (E) 5

Solution: Using the law of cosines in the triangle $\triangle ABC$, we obtain

$$\cos A = \frac{AB^2 + AC^2 - BC^2}{2(AB)(AC)} = \frac{13}{30}.$$

Using the law of cosines in the triangle $\triangle BAD$, and letting $AD = x$, we get the equation in x:

$$\frac{19}{30} = \frac{5^2 + x^2 - 5^2}{2(5)x} = \frac{x}{10},$$

which implies $x = \frac{19}{3}$ and $DC = \frac{8}{3}$. Hence, $\frac{AD}{DC} = \frac{19}{8}$ and so the answer is $\boxed{E}$. ■

10. [$*^5$] If x, y and z are positive numbers satisfying

$$\begin{cases} x + \frac{1}{y} = 2 \\ y + \frac{1}{z} = \frac{3}{4} \\ z + \frac{1}{x} = 14 \\ xyz > 1 \end{cases}.$$

What is the value of xyz?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: If we multiply all equalities term by term we obtain

$$xyz + x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xyz} = 21.$$

Subtracting all the given equations from this, we obtain $xyz + \frac{1}{xyz} = 4 + \frac{1}{4}$. This is a quadratic equation in xyz with obvious solutions 4 and 1/4. Hence, the answer is $\boxed{D}$. ■