Solutions to the Fifth Annual Columbus State Calculus Contest

April 16^{th}, 2017

1. *For two real values a and b the function*

$$f(x) = \begin{cases} (x+1)(x+2) & \text{if } x > 0 \\ \\ a\sin x + b\cos x, & \text{if } x \le 0, \end{cases}$$

is continuous and differentiable at 0. *What is $a - b$?*

(A) $\boxed{1}$ (B) 2 (C) 3 (D) 4 (E) 5

Solution: We need to have $2 = \lim\limits_{x \searrow 0} f(x) = f(0) = b$. Calculating the derivative for $x \neq 0$,

$$f'(x) = \begin{cases} 2x+3 & \text{if } x > 0 \\ \\ a\cos x - 2\sin x, & \text{if } x < 0. \end{cases}$$

Because f' has the Intermediate Value Property, we also need to have $3 = \lim\limits_{x \searrow 0} f'(x) = \lim\limits_{x \nearrow 0} f'(x) = a$. This implies $a - b = 1$. So, the answer is A. ■

2. *The functions $F(x) = \sin x$ and $G(x) = \cos x$ are defined for every real number x. Cauchy's theorem applied to F and G on the interval $[a, b]$, $0 < a < b < \pi$, gives*

$$\frac{F(b) - F(a)}{G(b) - G(a)} = \frac{F'(c)}{G'(c)}$$

with $c \in (a, b)$. For $a = \frac{9}{10}$ and $b = \frac{31}{10}$, what is $2c - 1$?

(A) 1 (B) 2 (C) $\boxed{3}$ (D) 4 (E) 5

Solution: The equality reduces to $\frac{\sin b - \sin a}{\cos b - \cos a} = -\cot c$ or $\frac{2\sin\frac{b-a}{2}\cos\frac{a+b}{2}}{-2\sin\frac{b-a}{2}\sin\frac{a+b}{2}} = -\cot c$. Hence, $\cot c = \cot\frac{a+b}{2}$ which shows that $c = \frac{a+b}{2}$. Therefore, we see that $c = (\frac{9}{10} + \frac{31}{10})/2 = 2$, and then the correct answer is C. ■

3. *The function*

$$g(x) = \left(\frac{x^2}{2} + x + 1\right)\cosh(x)$$

is defined by this rule for every real number x*. For every natural number* n*,* $g^{(n)}$ *denotes the* n^{th} *derivative of* g*. For how many values of* n *we have*

$$g^{(n)}(0) = 2017 \ ?$$

(A) 1 (B) $\boxed{2}$ (C) 3 (D) 4 (E) 5

Solution: Using the generalization of the product formula

$$(uv)^{(n)} = u^{(n)}v + nu^{(n-1)}v' + \frac{n(n-1)}{2}u^{(n-2)}v'' + ...$$

for $u(x) = \cosh(x)$ and $v(x) = \frac{x^2}{2} + x + 1$. We observe that $u^{(n)}(0) = 1$ if n is even, and $u^{(n)}(0) = 0$ if n is odd. Also, $v'(x) = x + 1$ and $v''(x) = 1$, which means $v^{(n)}(0) = 1$ for all $n = 0, 1, 2$ and $v^{(n)}(0) = 0$ if $n > 2$. Then, we get

$$g^{(n)}(0) = \begin{cases} 1 + \frac{n(n-1)}{2} \text{ if n is even,} \\ \\ n \text{ if n is odd.} \end{cases}$$

This gives $\boxed{B}$ as the correct answer, since $g^{(64)}(0) = g^{2017}(0) = 2017$ are the only writings of 2017 as desired. ■

4. *Suppose that* f *is defined on* $\mathbb{R}$ *by the rule*

$$f(x) = (1 - x)(1 + x^2).$$

The function is invertible and we denote its inverse by f^{-1}*. If* $h = f^{-1} \circ \ln \circ f$*, or in other words*

$$h(x) = f^{-1}(\ln(f(x))), \quad x < 1,$$

what is $3 + \frac{1}{h'(0)}$ *?*

(A) $\boxed{1}$ (B) 2 (C) 3 (D) 4 (E) 5

Solution: We observe that $f'(x) = 2x - 3x^2 - 1 = -2x^2 - (x - 1)^2 < 0$ which insures that f is strictly decreasing and so f^{-1} exists. Using the chain rule, we get $h' = (f^{-1})'(\ln f)\frac{f'}{f}$. So we need do compute

$$h'(0) = (f^{-1})'(\ln f(0))\frac{f'(0)}{f(0)}.$$

Because $f(0) = 1$, $f'(0) = -1$ and $f(1) = 0$, we can continue

$$h'(0) = -(f^{-1})'(0) = -\frac{1}{f'(f^{-1}(0))} = -\frac{1}{2}.$$

Ergo, the answer is $\boxed{A}$. ■

5. *The function*

$$G(x) = \frac{x+3}{(x^2+3)^2},$$

defined for all real values of x, has three distinct inflection points. One of them is at $x = 1$. What is the product of the other two values of x corresponding to the inflection points?

(A) 1 (B) 2 (C) $\boxed{3}$ (D) 4 (E) 5

Solution: We use the quotient rule to derive

$$G'(x) = \frac{(x^2+3)^2 - (x+3)2(x^2+3)(2x)}{(x^2+3)^4} = 3\frac{1-4x-x^2}{(x^2+3)^3}$$

and then

$$G''(x) = 3\frac{(-4-2x)(x^2+3)^3 - (1-4x-x^2)3(x^2+3)^2(2x)}{(x^2+3)^6} \Rightarrow$$

$$G''(x) = 3\frac{(-4-2x)(x^2+3) - (1-4x-x^2)3(2x)}{(x^2+3)^4} = 6\frac{3x^3+12x^2-3x-(x^3+2x^2+3x+6)}{(x^2+3)^4}.$$

Since $x = 1$ is a solution of $G''(x) = 0$, we look to factor $(x-1)$:

$$G''(x) = -6\frac{2x^3+10x^2-6x-6}{(x^2+3)^4} = -12\frac{(x-1)(x^2+6x+3)}{(x^2+3)^4}.$$

Hence, the answer is C. ■

6. *The function $g(x) = \cos(x) + 3\sin(2x)$ defined on the whole real line, has a maximum value of $\frac{m}{n}\sqrt{5}$, where $\frac{m}{n}$ is a rational number written in reduced form. What is the value of $(m+n)/2$?*

(A) 1 (B) 2 (C) 3 (D) $\boxed{4}$ (E) 5

Solution: We calculate the derivative of g,

$$g'(x) = -\sin x + 6\cos 2x = 6 - \sin x - 12\sin^2 x = (3 + 4\sin x)(3\sin x - 2).$$

For each of the critical points we obtain the values of the function $\pm(1-6\frac{3}{4})\frac{\sqrt{7}}{4} = \pm\frac{7\sqrt{7}}{8}$ and $\pm(1+6\frac{2}{3})\frac{\sqrt{5}}{3} = \pm\frac{5}{3}\sqrt{5}$.

Comparing the outputs we see that $\frac{5}{3}\sqrt{5}$ is the maximum value. Hence $(m+n)/2 = 4$ and the correct answer is D. ■

7. *For m a positive integer, we have*

$$L := \lim_{x\to\infty}\left[x^3 \ln\left(\frac{x+1}{x}\right) + \frac{x}{2} - x^2\right] = \frac{1}{m}.$$

What is m?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: We change the variable $x = \frac{1}{t}$ and obtain $L = \lim_{t\to 0}\frac{\ln(1+t)+t^2/2-t}{t^3}$. Using L'Hospital's Rule, we have

$$L = \lim_{t\to 0}\frac{\frac{1}{1+t} + t - 1}{3t^2} = \lim_{t\to 0}\frac{1}{3(1+t)} = \frac{1}{3}.$$

This implies the answer is C. ■

8. *The function h is defined for all real numbers and it is at least three times differentiable, satisfying*

$$h'''(x) + h''(x) + h'(x) + h(x) = 0$$

for all x. Knowing that $h(0) = 5$, $h'(0) = 1$ and $h''(0) = -3$, what is $h^{(2017)}(0)$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: The given identity shows that h is n-times differentiable for every n and $h^{(n)}(x) = -(h^{(n-1)}(0) + h^{(n-2)}(x) + h^{(n-3)}(x))$. If we denote by $a_n = h^{(n)}(0)$ we get a linear recurrence of order three $a_n = -(a_{n-1} + a_{n-2} + a_{n-3})$, $n \geq 3$. This show that $a_3 = -(-3+1+5) = -3$, $a_4 = -(-3-3+1) = 5$, $a_5 = -(5+(-3)+(-3)) = 1$, and $a_6 = -(1+5+(-3)) = -3$. We observe that at this point every value of a_n repeats going through a cycle of length 4. Therefore $a_{2017} = a_1 = 1$, so the answer is $\boxed{A}$. ■

9. *Consider the function $g(x) = \frac{-x}{x^2-4x+3}$ defined for all real numbers $x \in (1,3)$. For every natural number n, $g^{(n)}$ denotes the n^{th} derivative of g. Find*

$$\frac{g^{(2017)}(2)}{2017!}.$$

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: We write $g(x) = \frac{x-1-3(x-3)}{(x-1)(x-3)} = \frac{1}{x-3} - 3\frac{1}{x-1}$.
This implies that

$$g^{(n)}(x) = (-1)^n n! \left[\frac{1}{(x-3)^{n+1}} - 3\frac{1}{(x-1)^{n+1}}\right].$$

Hence $\frac{g^{(2017)}(2)}{2017!} = (-1)(1-3) = 2$. Then, he answer is B. ■

10. *The curve defined implicitly by*

$$x^3 + y^3 = 3xy + 3$$

passes through the point $(2, 1)$*. The tangent line to the curve at the point* $(2, 1)$ *intersects the curve at another point* (a, b)*. What is* $5a + 3b$ *?*

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: Using implicit differentiation, we have $3x^2 + 3y^2y' = 3(y + xy')$ which gives

$$y'(x) = \frac{x^2 - y}{x - y^2}. \tag{1}$$

This shows that $y'(2) = 3$ and so the tangent line has equation $y = 1+3(x-2)$ or $y = 3x-5$. Substituting this into the original equation gives $x^3+(3x-5)^3 = 3x(3x-5)+3$. This is a cubic equation which we know it has $x = 2$, as a root with multiplicity 2, so it factors as $4(7x - 8)(x - 2)^2 = 0$. Then the other intersection point is $(\frac{8}{7}, b)$ where $b = 3a - 5$. This gives $5a + 3b = 5a + 3(3a - 5) = 14a - 15 = 1$. Therefore the answer is A. ■

11. *The curve defined implicitly by*

$$x^3 + y^3 = 3xy + 3$$

passes through the point $(2, 1)$*. Find* $y''(2)/4$.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: If we use the derivative we got in (1) we get

$$y''(x) = \frac{(2x - y')(x - y^2) - (x^2 - y)(1 - 2yy')}{(x - y^2)^2} \Rightarrow$$

$$y''(2) = \frac{(4-3)(2-1) - 3(1-6)}{(2-1)^2} = 16$$

Hence, the answer is D. ■

12. *The equation of the tangent line to the graph of equation*

$$(y+1)\ln(2x-3) - (x-5)\ln(3y-2) = 0$$

at the point $(2, 1)$ *passes through the point* $(-7, \omega)$. *What is* ω*?*

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: Using implicit differentiation we obtain

$$y'\ln(2x-3) + \frac{2(y+1)}{2x-3} - \ln(3y-2) - \frac{(x-5)3y'}{3y-2} = 0 \Rightarrow$$

$$2(2) + 3(3y') = 0 \Rightarrow y' = -4/9.$$

Then the equation of the tangent line is $y = 1 - 4(x-2)/9$. As a result, $\omega = 1 - 4(-7-2)/9 = 5$. Thus, the correct answer is E. ■

13. If

$$F(x) = \int_x^{4x} \frac{1}{3 + (4-t)^2(\ln t)^2} dt$$

for $x > 0$, what is $F'(1)$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: Using the Fundamental Theorem of Calculus,

$$F'(x) = \frac{4}{3 + (4-4x)^2(\ln 4x)^2} - \frac{1}{3 + (4-x)^2(\ln x)^2} \Rightarrow$$

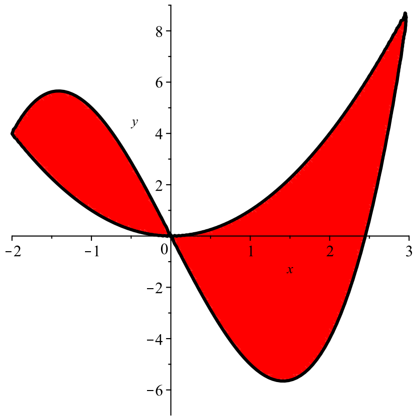
$$F'(1) = \frac{4}{3} - \frac{1}{3} = 1.$$

This shows that A is the correct answer. ■

14. [$*^1$] *The graphs of $y = x^2$ and $y = x^3 - 6x$ for $x \in [-2, 3]$ are shown in the figure on the right. If A is the area between their graphs in this interval (shaded in red), then*

$$A = \frac{m}{n}$$

is a rational number written in reduced form. What is $m - 21n$?



(A) 1 (B) 2 (C) 3

(D) 4 (E) 5

Solution: The graphs intersect at $a = -2$, $x = 0$ and $x = 3$. We have $A = \int_{-2}^{3} |x^3 - 6x - x^2| dx$. If $G(x) = \frac{x^4}{4} - \frac{x^3}{3} - 3x^2$ we see that $A = |G(3) + G(-2) - 2G(0)| = \frac{253}{12}$. Thus, $m - 21n = 253 - 21(12) = 1$. Hence the answer is A. ■

15. [$*^1$] *We consider the quadratic function $G(x) = 2x(1 - x)$ defined over the interval $I := [0, 1]$ with values in the interval I. For a positive integer n, we denote $G_n = \underbrace{G \circ G \circ ... \circ G}_{n \text{ times}}$. Knowing that*

$$\int_0^1 G_{2017}(x)dx = \frac{m}{n}$$

is in reduced terms, what is $n - 2m$?

(A) $\boxed{1}$ (B) 2 (C) 3 (D) 4 (E) 5

Solution: We observe that

$$G(x) = 2(x - x^2) = 2[\frac{1}{4} - (\frac{1}{2} - x)^2] = \frac{1}{2} - 2(\frac{1}{2} - x)^2.$$

This shows that

$$(G \circ G)(x) = \frac{1}{2} - 2(\frac{1}{2} - G(x))^2 = \frac{1}{2} - 2^3(\frac{1}{2} - x)^4 \Rightarrow$$

$$(G \circ G \circ G)(x) = \frac{1}{2} - 2(\frac{1}{2} - (G \circ G)(x))^2 = \frac{1}{2} - 2^7(\frac{1}{2} - x)^8 \Rightarrow$$

$$G_n(x) = \frac{1}{2} - 2^{2^n - 1}(\frac{1}{2} - x)^{2^n}.$$

Then $\int_0^1 G_n(x)dx = \frac{1}{2} - 2^{2^n-1}\int_{-1/2}^{1/2} t^m dt$ where $k = 2^n$. Hence,

$$\int_0^1 G_n(x)dx = \frac{1}{2} - 2^{2^n-1}\frac{2(1/2)^{k+1}}{k+1} = \frac{k}{2(k+1)} = \frac{2^{n-1}}{2^n+1}.$$

Hence, the answer is A. ■

16. *The rational number $p = \frac{m}{n}$ (in reduced form) has the property that*

$$L := \lim_{x\to\infty} x^p(\sqrt[3]{x+1} + \sqrt[3]{x-1} - 2\sqrt[3]{x})$$

is some non-zero real number. What is $m-n$?

(A) 1 (B) $\boxed{2}$ (C) 3 (D) 4 (E) 5

Solution: Substituting $x = \frac{1}{t}$ we obtain

$$L = \lim_{t\to 0}\frac{\sqrt[3]{1+t} + \sqrt[3]{1-t} - 2}{t^{p+\frac{1}{3}}}.$$

This gives that $k = p + \frac{1}{3}$ must be at least one and we can use L'Hospital's Rule:

$$L = \lim_{t\to 0}\frac{\frac{1}{3}[(1+t)^{-2/3} - (1-t)^{-2/3}]}{kt^{k-1}}.$$

At this point it is clear that $k-1$ must be equal to 1 ($k = 2$, $p = \frac{5}{3}$) and one can use L'Hospital's Rule again:

$$L = \lim_{t\to 0}\frac{\frac{1}{3}(-\frac{2}{3})[(1+t)^{-5/3} + (1-t)^{-5/3}]}{2} = -\frac{2}{9} \neq 0.$$

So, B is the correct answer. ■

17. *The recurrent sequence $\{x_n\}$ satisfies the recurrence $x_{n+1} = \frac{6x_n}{1+x_n}$ for every $n \geq 1$ and $x_1 = 1/2017$. Knowing that $\{x_n\}$ is convergent to L, what is L?*

(A) 1 (B) 2 (C) 3 (D) 4 (E) $\boxed{5}$

Solution: Since x_{n+1} converges to L, we have the equation in L: $L = \frac{6L}{1+L}$. We either get $L = 0$ or $L = 5$. Then the answer is E. ■

18. *We define f by the rule $f(x) = 5(\sin x)^4 + 3(\cos x)^4$ for all real numbers x. Knowing that c is the smallest positive number with the property*

$$f(c) = \frac{1}{\pi}\int_0^{\pi} f(x)dx$$

find $\frac{\pi}{c}$.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: We can simplify f in the following way

$$f(x) = \frac{5}{4}(1-\cos 2x)^2 + \frac{3}{4}(1+\cos 2x)^2 = 2 - \cos 2x + 1 + \cos 4x, \Leftrightarrow$$

$$f(x) = 3 - \cos 2x + 1 + \cos 4x.$$

So, $\int_0^\pi f(x)dx = 3\pi$. Then the given equation in c is equivalent to $\cos(4c) - \cos(2c) = 0$. Using the double angle formula again, this turns into

$$2\cos^2 2c - \cos 2c - 1 = (2\cos 2c + 1)(\cos 2c - 1) = 0.$$

The smallest positive solution of this equation is clearly given by $2c = 2\pi/3$, which attracts $\pi/c = 3$. Hence, $\boxed{C}$ is the answer. ■

19. [$*^2$] *We have for some natural number* m

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2+k^2} = \frac{\ln m}{m}$$

Find m.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: We use the Riemann Sums definition of the definite integral for $f(x) = \frac{x}{1+x^2}$ on the interval $[0,1]$. We can compute easily $\int_0^2 f(x)dx = \frac{1}{2}\ln(x^2+1)|_0^1 = \frac{\ln 2}{2}$. Hence

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2+k^2} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k/n}{1+(k/n)^2} = \frac{\ln 2}{2}.$$

This shows that the correct answer is B. ■

20. [$*^1$] The function f is defined for $x \in [0, \frac{1}{e}]$ in the following way: for every $x \in [0, \frac{1}{e}]$, $f(x)$ is the solution of the equation

$$x = f(x)e^{-f(x)}.$$

Given that

$$\int_0^{1/e} f(x)dx = \frac{m}{e} - n,$$

where m and n are integers, what is m ?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: The function f is the inverse of the function $g(t) = te^{-t}$ on the interval $[0, 1]$. Then,

$$\int_0^{1/e} f(x)dx + \int_0^1 g(t)dt = 1/e$$

Since, $\int_0^1 g(t)dt = -e^{-t}(t+1)|_0^1 = 1 - \frac{2}{e}$ we obtain that $\int_0^{1/e} f(x)dx = \frac{3}{e} - 1$.

Thus, C is the correct answer. ■