Fifth Annual Columbus State Calculus Contest-Precalculus Test

Sponsored by
The Columbus State University
Department of Mathematics
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The Columbus State University Mathematics faculty welcome you to this year's Pre-Calculus/Calculus contest. We wish you success on this test and in your future studies.

Instructions

This is a 120-minute, 10-problem, multiple choice examination. There are five possible responses to each question. You should select the one "*best*" answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with a No. 2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response.

Throughout the exam, $\overline{AB}$ will denote the line segment from point A to point B and AB will denote the length of $\overline{AB}$. Pre-drawn geometric figures are not necessarily drawn to scale. The measure of the angle $\angle ABC$ is denoted by $m\angle ABC$.

The examination will be scored on the basis of $+23$ *for each correct answer,* -3 *for each incorrect selection, and* 0 *for each omitted item.*

No phones or any communication devices can be used. Calculators with CAS such as the TI-89 are not allowed. In fact, the test is designed in such a way that you do not really need a calculator. The problems denoted with $[\star^n]$ are tie-breaker problems, so more attention should be given to them. Possibly, include written justification for your answers, on the pages provided at the end of the test, especially for the tie-breaker problems. It is not necessary, but you may find useful reading the "Theoretical facts" part.

Do not open your test until instructed to do so!

Theoretical facts that you may find useful.

Theorem 1: (**Factor Theorem**) Given a polynomial P, then P is divisible by $x-a$ if and only if $P(a)=0$. The remainder of the division of $P(x)$ by $x-a$ is $P(a)$.

Trigonometric formulae:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta, \quad \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\cos a - \cos b = 2\sin\frac{b-a}{2}\sin\frac{a+b}{2} \quad \text{and} \quad \cos a + \cos b = 2\cos\frac{a-b}{2}\cos\frac{a+b}{2}$$

Double Angle Formulae: $\sin 2\alpha = 2\sin\alpha\cos\alpha$, $\cos 2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$.

Theorem 3: A polynomial $P(x) = a_0 + a_1 x + \cdots a_n x^n$ has a rational zero, p/q (reduced form), if p divides a_n and q divides a_0.

Theorem 4: Quadratic formula and Viete's relations: the equation $ax^2+bx+c=0$ has two zeros (x_1, x_2, real or complex) given by

$$\boxed{x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}}.$$

These zeros satisfy Viete's relations:

$$\boxed{x_1 + x_2 = -\frac{b}{a}, \quad x_1 x_2 = \frac{c}{a}}.$$

Theorem 5: Heron's formula: The area of a triangle is given by the formulae

$$A = \sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{4}\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}$$

where $s = \frac{a+b+c}{2}$ and a, b and c are the sides of the triangle.

Theorem 6: The Law of Sines in a triangle states that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin A}$$

where A, B and C are the measurement of the interior angles of the triangle.

Theorem 7: (**Pythagorean Theorem:**) In a right triangle with legs b, c and hypothenuse a, we have $b^2+c^2=a^2$.

Theorem 8: Properties of logarithms:

$$\log_a u = v \Leftrightarrow u = a^v$$

$$\log_a u + \log_a v = \log_a uv, \quad \log_a u^v = v\log_a u$$

Pre-calculus Problems

1. For two unique positive integers a and b, we have $2017 = a^2 + b^4$. Also, a can be written as

$$c_0 + c_1 b + c_2 b^2 + c_3 b^3 + c_4 b^4$$

where c_0, c_1, c_2, c_3 and c_4 are either 1, 0 or -1. What is the value of

$$c_4 - c_3 - c_2 - c_1 - c_0?$$

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

2. For a and b positive integers, with a not a perfect square, it is given that $\sqrt{a} - b$ is a root of the quadratic equation $x^2 + ax - b = 0$ and $\sqrt{a} + b$ is a root of the quadratic equation $x^2 - ax - b = 0$, what is $a + b$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

3. [$*^1$] Three angles α, β and γ satisfy the equalities $\tan\alpha = x_1$, $\tan\beta = x_2$ and $\tan\gamma = x_3$, where x_1, x_2 and x_3 are the roots of the cubic equation $5x^3 - 14x^2 + 2x + 1 = 0$. What is the value of

$$\tan(\alpha + \beta + \gamma)?$$

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

4. It is known that 2017 is the 306^{th} prime and it can be written as a sum of two triangular numbers (numbers of the form 1, $1+2$, $1+2+3$, etc.) in a unique way. For positive integers m, n, and k, with $n > m$, we have

$$\sum_{j=m+1}^{n} j^3 = (m+1)^3 + (m+2)^3 + ... + n^3 = 2017(2017 - k).$$

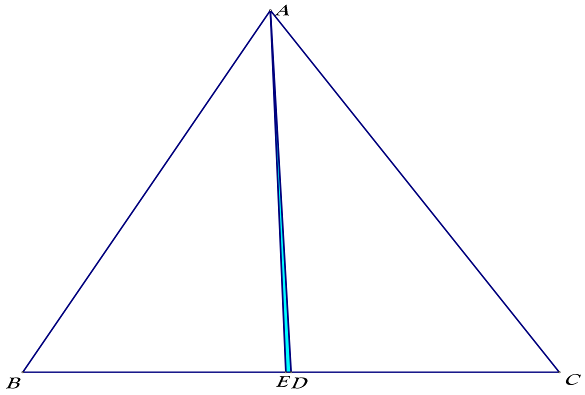
Knowing that $m + n$ is the smallest number with this property, find k.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

5. In the triangle ABC, $AB = 51$, $BC = 52$, $AC = 53$. Let D denote the midpoint of $\overline{BC}$ and let E denote the intersection of $\overline{BC}$ with the bisector of angle $\angle BAC$. If the area of the triangle AED (in square units) is equal to $\frac{15x}{4}$ what is x?

(A) 1 (B) 2 (C) 3

(D) 4 (E) 5

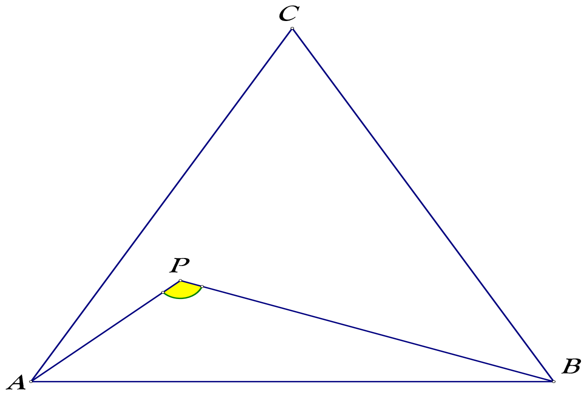


6. From the interior of an equilateral triangle ABC one takes a random point P. The probability that the angle $\angle APB$ has measure more than $120°$ is equal to $\frac{m}{n}\pi\sqrt{3}-\frac{1}{3}$, where the $\frac{m}{n}$ is a positive rational number written in reduced form. What is the value of

$$7m - n\ ?$$

(A) 1 (B) 2 (C) 3

(D) 4 (E) 5



7. [$*^2$] The number of ordered pairs of positive real numbers (u, v) such that

$$(u + iv)^{2017} = u - iv$$

is a number of three digits abc (written in base 10). What is $a - b$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

8. [$*^3$] Positive integers a, b, and c are chosen so that $a < b < c$, and the system of equations

$$2x + y = 2017 \quad \text{and} \quad y = |x - a| + |x - b| + |x - c|$$

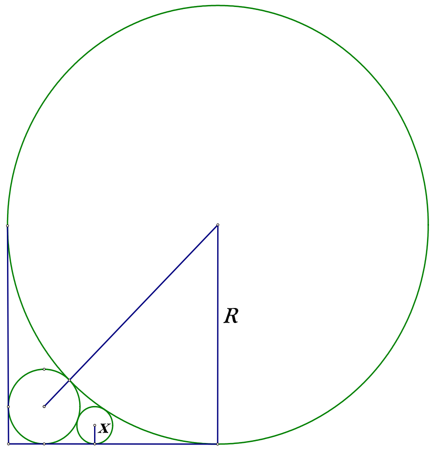
has exactly one solution (x, y). The minimum value of c possible is a number of four digits $d_1d_2d_3d_4$ (written in base 10). What is $d_4 + 2d_3 + 3d_2 - 5d_1$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

9. [$*^4$] Two circles of radii R and r $(R > r)$ are tangent to one another in such a way their common tangent lines are perpendicular (see the accompanying figure). Another circle of radius x is tangent to one of these tangent lines and tangent to both of the circles. The ratio $\frac{R}{2x}$ can be written as $m + n\sqrt{2}$ for two positive integers m and n. What is $m+n$?

(A) 1 (B) 2 (C) 3

(D) 4 (E) 5



10. [$*^5$] Three real numbers x, y, and z satisfy

$$\begin{cases} \log_2(x+y) = z \\ \log_2(x^2+y^2) = z+1 \\ 2x+y = 6 \\ xy > 4 \end{cases} .$$

What is the closest value to z?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5