Solutions to the Fifth Annual Columbus State Calculus Pre-calculus contest

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1. *For two unique positive integers a and b, we have* $2017 = a^2 + b^4$*. Also, a can be written as*

$$c_0 + c_1 b + c_2 b^2 + c_3 b^3 + c_4 b^4$$

where c_0, c_1, c_2, c_3 *and* c_4 *are either* 1, 0 *or* −1*. What is the value of*

$$c_4 - c_3 - c_2 - c_1 - c_0?$$

(A) 1 (B) 2 (C) 3 (D) $\boxed{4}$ (E) 5

Solution: For b from 1 to 6 we observe that $2017 - b^4$ is 2016, 2001, 1936, 1761, 1392, and 721 respectively. Only 1936 is a perfect square, which is 44^2. Hence, $a = 44$ and then

$$44 = -1 - 0(3^1) - 1(3^2) - 1(3^3) + 1(3^4)$$

implies that $c_4 - c_3 - c_2 - c_1 - c_0 = 4$. Therefore, the correct answer is D. ■

2. *For a and b positive integers, with a not a perfect square, it is given that* $\sqrt{a} - b$ *is a root of the quadratic equation* $x^2 + ax - b = 0$ *and* $\sqrt{a} + b$ *is a root of the quadratic equation* $x^2 - ax - b = 0$*, what is* $a + b$*?*

(A) 1 (B) 2 (C) $\boxed{3}$ (D) 4 (E) 5

Solution: The information given is sufficient to conclude that $\pm\sqrt{a} - b$ are the roots of $x^2 + ax - b = 0$. From Viete's relations, we obtain that $-2b = -a$ and $b^2 - a = -b$. This implies $b^2 = b$ which gives $a = 2$ and $b = 1$. So, $a + b = 3$ which means C is the correct choice here. ■

3. [$*^1$] *Three angles* α, β *and* γ *satisfy the equalities* $\tan\alpha = x_1$, $\tan\beta = x_2$ *and* $\tan\gamma = x_3$*, where* x_1, x_2 *and* x_3 *are the roots of the cubic equation* $5x^3 - 14x^2 + 2x + 1 = 0$*. What is the value of*

$$\tan(\alpha + \beta + \gamma)?$$

(A) 1 (B) 2 (C) 3 (D) 4 (E) $\boxed{5}$

Solution: (Method I) We have, for two angles $\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$ which implies that

$$\tan(\alpha+\beta+\gamma) = \frac{\frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)} + \tan(\gamma)}{1 - \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}\tan(\gamma)} = \frac{x_1+x_2+x_3-x_1x_2x_3}{1-x_1x_2-x_2x_3-x_3x_1} \Rightarrow$$

$$\tan(\alpha+\beta+\gamma) = \frac{\frac{14}{5} - \frac{-1}{5}}{1-\frac{2}{5}} = \frac{15}{5}\frac{5}{3} = 5.$$

This shows that the answer is $\boxed{E}$. ■

(Method II) We observe that $5x^3 - 14x^2 + 2x + 1 = (5x+1)(x^2 - 3x + 1)$ which means that one of the solutions is $-1/5$ and the other two have a product of 1. Let's say $\tan(\alpha) = -1/5$ and then $\tan\beta\tan\gamma = 1$. The last equality implies $\cos(\beta+\gamma) = 0$. Therefore $\beta+\gamma = \frac{\pi}{2} + k\pi$ which implies that

$$\tan(\alpha+\beta+\gamma) = \tan[\frac{\pi}{2} - (-\alpha)] = \cot(-\alpha) = 5. \ ■$$

4. *It is known that* 2017 *is the* 306^{th} *prime and it can be written as a sum of two triangular numbers (numbers of the form* 1, $1+2$, $1+2+3$, *etc.) in a unique way. For positive integers* m, n, *and* k, *with* $n > m$, *we have*

$$\sum_{j=m+1}^{n} j^3 = (m+1)^3 + (m+2)^3 + ... + n^3 = 2017(2017-k).$$

Knowing that $m+n$ *is the smallest number with this property, find* k.

(A) 1 (B) $\boxed{2}$ (C) 3 (D) 4 (E) 5

Solution: We have the known formula $\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$, which implies

$$\sum_{j=m+1}^{n} j^3 = \frac{n^2(n+1)^2}{4} - \frac{m^2(m+1)^2}{4} = \frac{1}{4}(n^2+n-m^2-m)(n^2+m^2+m+n) \Leftrightarrow$$

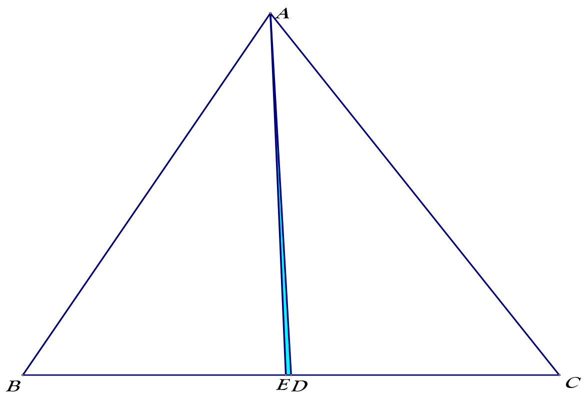
$$2017(2017-k) = \frac{1}{2}(n-m)(m+n+1)(\frac{n(n+1)}{2} + \frac{m(m+1)}{2}).$$

Since it is possible that $2017 = \frac{n(n+1)}{2} + \frac{m(m+1)}{2} = \frac{63(64)}{2} + \frac{1(2)}{2}$ this gives the smallest possibility for $m+n = 63+1 = 64$ (2017 dividing $n-m$ implies $n+m > 2017$, and 2017 dividing $m+n+1$ gives $m+n > 2016$). Hence $\frac{1}{2}(n-m)(n+m+1) = 31(65) = 2015$ which shows that $k = 2$. Hence, $\boxed{B}$ is the correct answer. ■

5. *In the triangle ABC, $AB = 51$, $BC = 52$, $AC = 53$. Let D denote the midpoint of $\overline{BC}$ and let E denote the intersection of $\overline{BC}$ with the bisector of angle $\angle BAC$. If the area of the triangle AED (in square units) is equal to $\frac{15x}{4}$ what is x?*

(A) 1 (B) 2 (C) $\boxed{3}$

(D) 4 (E) 5



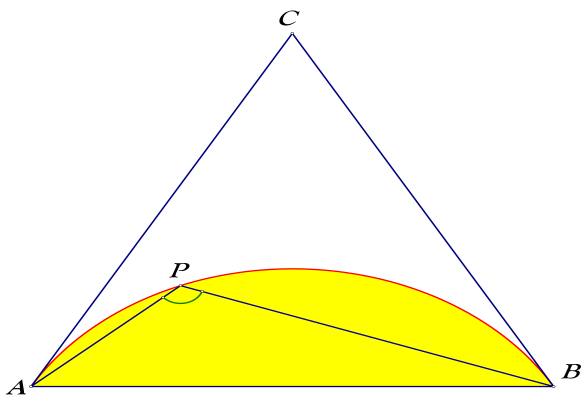
Solution: We compute the area of $\triangle ABC$ first, using Heron's formula $[ABC] = \sqrt{78(78-51)(78-52)(78-53)} = \sqrt{2(3)(3)(9)(2)(13)(25)} = 2(9)(13)(5) = 1170$. Then the area of $\triangle ABD = 1170/2 = 585$. Also, this means that the altitude corresponding to A is equal to $h = 2[ABC]/BC = 2(1170)/52 = 1170/(2(2)(13)) = 45$. Using the Angle Bisector Theorem we have $\frac{AB}{AC} = \frac{BE}{EC}$ or $\frac{51}{53} = \frac{BE}{EC}$. Therefore, $\frac{51}{53+51} = \frac{BE}{BE+EC}$ or $\frac{51}{2(52)} = \frac{BE}{52}$ which gives $BE = \frac{1}{2}$. So, $DE = BE - BD = 26 - \frac{51}{2} = \frac{1}{2}$. Then, the area of $\triangle ADE = \frac{DE(h)}{2} = \frac{45}{4}$. Therefore, $x = 3$ and so the answer is $\boxed{C}$. ■

6. *From the interior of an equilateral triangle ABC one takes a random point P. The probability that the angle $\angle APB$ has measure more than $120°$ is equal to $\frac{m}{n}\pi\sqrt{3} - \frac{1}{3}$, where the $\frac{m}{n}$ is a positive rational number written in reduced form. What is the value of*

$$7m - n\ ?$$

(A) 1 (B) 2 (C) 3

(D) 4 (E) 5



Solution: We construct the arc corresponding to the points P' for which $m(\angle AP'B) = 120°$. This is an arc of a circle tangent to the sides $\overline{AC}$ and $\overline{BC}$ (by the converse of the angle inscribe theorem). Then the point P must be inside of the arc segment determined by this arc. So, we need to compute the area of this arc segment, say $\mathcal{A}$ and then the required probability is $\frac{\mathcal{A}}{[ABC]}$. The radius of the circle containing this arc is $R = \ell \tan 30^\circ = \frac{\sqrt{3}}{3}\ell$, where ℓ is the side-length of the equilateral triangle ABC. Then

$$\mathcal{A} = \frac{1}{3}(\pi R^2 - [ABC]) = \frac{1}{3}(\frac{\pi}{3}\ell^2 - \frac{\ell^2\sqrt{3}}{4}),$$

which implies that the probability is

$$P = \frac{\mathcal{A}}{[ABC]} = \frac{4\pi\sqrt{3}}{27} - \frac{1}{3}.$$

Hence, $7m - n = 7(4) - 27 = 1$, and the correct answer is $\boxed{A}$. ■

7. [$*^2$] The number of ordered pairs of positive real numbers (u, v) such that

$$(u + iv)^{2017} = u - iv$$

is a number of three digits abc (written in base 10). What is $a - b$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: If we denote by $z = u + iv$ we see that $|z|^{2017} = |z|$. Because $z \neq 0$, we conclude that $|z| = 1$. Hence, $\overline{z} = \frac{1}{z}$, and the given equation becomes $z^{2018} = 1$. The solutions of this equation, as complex numbers, from the set of all roots of unity of order 2018 (vertices of a regular polygon). Two of these are 1 and -1. There are no such units on the y-axis, because 2018 is not divisible by 4. Then, there are exactly $(2018 - 2)/4 = 504$ inside of the first quadrant. Hence, $a - b = 5 - 0 = 5$ that the correct answer is $\boxed{E}$. ■

8. [$*^3$] Positive integers a, b, and c are chosen so that $a < b < c$, and the system of equations

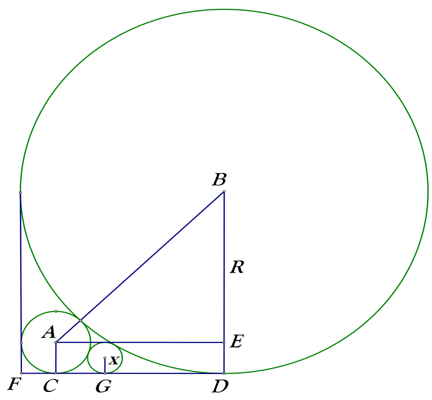
$$2x + y = 2017 \quad \text{and} \quad y = |x - a| + |x - b| + |x - c|$$

has exactly one solution (x, y). The minimum value of c possible is a number of four digits $d_1d_2d_3d_4$ (written in base 10). What is $d_4 + 2d_3 + 3d_2 - 5d_1$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: Eliminating y, we get $2017 = f(x) = 2x + |x - a| + |x - b| + |x - c|$. The slope of the function f on the interval $(-\infty, a)$ is -1, on the interval (a, b) is 1, on the interval (b, c) is 3 and on the interval (c, ∞) is 5. Since f is a continuous function, f has a unique minimum at $x = a$ which is equal to $f(a) = 2a + b - a + c - a = b + c$. Then, this minimum must be equal to 2017. Indeed, if $b + c > 2017$ the system has no solutions, and if $b + c < 2017$ then we have two distinct solutions. Because $b \leq c - 1$, we obtain that $2017 = b + c \leq 2c - 1$. This attracts $c \geq 2018/2 = 1009$. One can check that 1009 is indeed the minimum value of c with the required properties. So, $d_4 + 2d_3 + 3d_2 - 5d_1 = 9 - 5 = 4$ and the correct answer is $\boxed{D}$. ■

9. [$*^4$] Two circles of radii R and r ($R > r$) are tangent to one another in such a way their common tangent lines are perpendicular (see the accompanying figure). Another circle of radius x is tangent to one of these tangent lines and tangent to both of the circles. The ratio $\frac{R}{2x}$ can be written as $m + n\sqrt{2}$ for two positive integers m and n. What is $m+n$?



(A) 1 (B) 2 (C) 3

(D) 4 (E) 5

Solution: In the accompanying figure, we see that $AE^2 + BE^2 = AB^2$ or $CD = \sqrt{(R+r)^2 - (R-r)^2} = 2\sqrt{Rr}$. Because the two tangent lines are perpendicular, the triangle ABE is a right isosceles triangle. This gives $AE = BE$ or $2\sqrt{Rr} = R - r$. Solving for R we obtain $R = (3 + 2\sqrt{2})r = (\sqrt{2}+1)^2 r$. Then the equation for x is

$$CD = 2\sqrt{Rr} = CG + GD = 2\sqrt{rx} + 2\sqrt{xR} \Rightarrow$$

$$\sqrt{x}[\sqrt{r} + (\sqrt{2}+1)\sqrt{r}] = (\sqrt{2}+1)r \Rightarrow$$

$$\sqrt{x} = \frac{\sqrt{r}}{\sqrt{2}} \Rightarrow \frac{R}{2x} = 3 + 2\sqrt{2}.$$

Hence the answer is $\boxed{E}$. ■

10. [$*^5$] Three real numbers x, y, and z satisfy

$$\begin{cases} \log_2(x+y) = z \\ \log_2(x^2+y^2) = z+1 \\ 2x + y = 6 \\ xy > 4 \end{cases}.$$

What is the closest value to z?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: We observe that $x+y=2^z$ and $x^2+y^2=2^{z+1}=2(x+y)$. Since $y=6-2x$, we obtain a quadratic equation in x: $x^2+(6-2x)^2=(x+6-2x)$. This reduces to $0=5x^2-22x+24=(x-2)(5x-12)$. If $x=2$, then $y=6-2x=2$ which does not satisfy the last inequality of our problem. It remains that $x=12/5$ and so $y=6-2x=6/5$. Then $z=\log_2(x+y)=\log_2(18/5)\approx 1.84$. Without using a calculator, we have

$$\log_2(18/5) < \log_2(20/5) = \log_2(4) = 2$$

and

$$\log_2(18/5) > 3/2 \Leftrightarrow 18/5 > 2\sqrt{2} \Leftrightarrow 81/25 > 2(true).$$

Hence, the answer is $\boxed{B}$. ∎