Thirty–Eighth Annual Columbus State Invitational Mathematics Tournament

Sponsored by

The Columbus State University

Department of Mathematics

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The Columbus State University Mathematics faculty welcome you to this year's tournament and to our campus. We wish you success on this test and in your future studies.

Instructions

This is a 90-minute, 50-problem, multiple choice examination. There are five possible responses to each question. You should select the one "best" answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with a No. 2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response. The examination will be scored on the basis of +12 for each correct answer, −3 for each incorrect selection, and 0 for each omitted item. Each student will be given an initial score of +200.

Pre-selected problems will be used as tie-breakers for individual awards. These problems, designated with an asterisk (*), in order of consideration are: 1, 4, 8, 9, 10, 12, 14, 15, 18, 20, 21, 23, 25, 27, 28, 33, 34, 35, 36, 40, 42, 44, 48, 49, 50.

Throughout the exam, $\overline{AB}$ will denote the line segment from point A to point B and AB will denote the length of $\overline{AB}$. Pre-drawn geometric figures are not necessarily drawn to scale. The measure of the angle $\angle ABC$ is denoted by $m\angle ABC$.

Review and check your score sheet carefully. **Your student identification number and your school number must be encoded correctly on your score sheet.**

When you complete your test, bring your pencil, scratch paper and answer sheet to the test monitor. Leave the room after you have handed in your answer sheet. Please leave quietly so as not to disturb the other contestants. Do not congregate outside the doors by the testing area. You may keep your copy of the test. Your sponsor will have a copy of solutions to the test problems.

Do not open your test until instructed to do so!

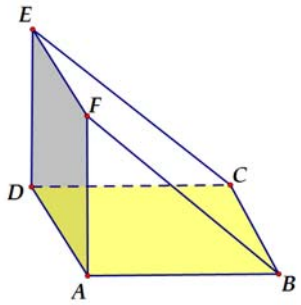
1. * Consider the experiment of throwing two 6–sided fair dice, where, the faces are numbered from 1 to 6. What is the probability of the event that the sum of the values of the two dice is 7?

(a) $\frac{1}{12}$ (b) $\frac{1}{9}$ (c) $\frac{1}{6}$ (d) $\frac{2}{9}$ (e) $\frac{1}{4}$

2. Find the last digit of the sum $0! + 2! + 4! + \cdots + 2010! + 2012!$.

(a) 1 (b) 3 (c) 5 (d) 7 (e) 9

3. Squares *ABCD* and *ADEF* are in perpendicular planes. If $AB = 4$, find *FC*.



(a) $2\sqrt{3}$ (b) $4\sqrt{2}$ (c) $3\sqrt{5}$ (d) $4\sqrt{3}$ (e) $2\sqrt{5}$

4. * The probability distribution of your winnings at a casino's card game is shown below.

X	$0	$5	$10	$25
P(X)	0.1	0.4	0.3	0.2

How much should you expect to win if you play the game once?

(a) $5 (b) $7.5 (c) $10 (d) $12.5 (e) $15

5. Working alone, Al can wash a hotel restaurant's dishes in three hours. Jenny can do the same job in six hours. If they worked together, how long would it take them to complete this task?

(a) 9 hours (b) $\frac{9}{2}$ hours (c) 2 hours (d) $\frac{2}{9}$ hours (e) $\frac{1}{9}$ hours

6. What is the sum of all positive integer divisors of 2012?

(a) 1034 (b) 1724 (c) 2347 (d) 3528 (e) 5213

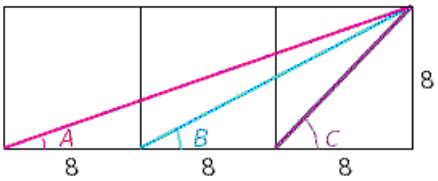
7. How many roots of the equation $1-15x+70x^2-120x^3+64x^4=0$ lie in the interval $[0,1]$?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

8. * Polynomials P and Q satisfy the equation $P(x-2)=(x^2+1)\cdot Q(x-1)-x-1$ for all real numbers x. When $P(x)$ is divided by $(x-3)$, the remainder is 20. Determine the remainder when $Q(x)$ is divided by $(x-4)$.

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

9. * The following is a rectangle made of three congruent squares of side length 8 units. Determine $m\angle A+m\angle B+m\angle C$.



(a) 67.5° (b) 78.75° (c) 90° (d) 101.25° (e) 112.5°

10. * The rational function $f(x)=\dfrac{x^3-8}{x^2-4}$ is defined for all real numbers except for $x=\pm 2$. Given that it has a vertical asymptote $x=a$ and a slant asymptote $y=mx+b$, determine the sum $a+m+b$.

(a) -3 (b) -1 (c) 0 (d) 1 (e) 3

11. If $N=2^{6n+2}+4^{3n+2}+8^{2n+1}$, what is the smallest positive integer x such that the product xN is a perfect square for all positive integers n?

(a) 2 (b) 3 (c) 5 (d) 7 (e) 11

12. * Let $A=\{1,3,4\}$ and $B=\{1,2,3,4,5,6,7\}$. How many subsets of B contain A?

(a) 16 (b) 32 (c) 48 (d) 96 (e) 112

13. Determine the value of the product $(\tan 1°)(\tan 2°)(\tan 3°)\ldots(\tan 88°)(\tan 89°)$.

(a) 0 (b) 1 (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$ (e) $\frac{\pi}{6}$

14. * Determine the solution set of the equation $\log_3\left(9 \cdot 3^{x+3}\right) = 3|x| + 1$.

(a) $\{-1, 2\}$ (b) $\{0,2\}$ (c) $\{0,3\}$ (d) $\{4,0\}$ (e) $\{2,3\}$

15. * Quadrilateral *ABCD* is inscribed in a circle in such a way that exactly one pair of opposite sides intercept congruent arcs. Being as specific as possible, what type of quadrilateral is *ABCD*?

(a) Square (b) Rectangle (c) Kite (d) Trapezoid (e) Parallelogram

16. Let $x = \sqrt{7+\sqrt{13}} - \sqrt{7-\sqrt{13}}$. Evaluate the expression $\left(x^3 - 2x - 2\right)^{2012}$.

(a) 2^{8048} (b) 2^{2024} (c) 4^{1006} (d) 4^{503} (e) 8^{6036}

17. In space, which of the following propositions is false?

(a) A line parallel to one of two parallel lines is also parallel to the other line.
(b) Three parallel lines may not be coplanar.
(c) A line intersecting one of two parallel lines must intersect the other line as well.
(d) Through a point not on a given plane, there is exactly one plane parallel to the given plane.
(e) Through two distinct points in a plane, there is exactly one plane perpendicular to that plane.

18. * In the experiment of selecting 5 marbles without replacement from a bag containing 6 yellow and 4 red marbles, what is the probability of getting 3 yellow and 2 red marbles?

(a) $\frac{1}{3}$ (b) $\frac{2}{5}$ (c) $\frac{10}{21}$ (d) $\frac{1}{2}$ (e) $\frac{5}{9}$

19. What is the product of all integers *n* such that $\sqrt{n^2 - 37}$ is an integer?

(a) -225 (b) -324 (c) -361 (d) -441 (e) -576

20. * How many real solutions does the equation $\left(x^2-2\right)^{x^2+2x}=1$ have?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

21. * Let $E_n=\sum_{k=0}^{n}\binom{n}{k}$ with n a natural number. Evaluate the difference $E_{n+1}-E_n$.

(a) 2^n-1 (b) 2^{n-1} (c) 2^n (d) 2^{n+1} (e) 2^n+1

22. Evaluate the expression $\sum_{n=1}^{10}\sum_{m=2}^{8}(mn-3n)$.

(a) -2012 (b) -1006 (c) 0 (d) 770 (e) 2012

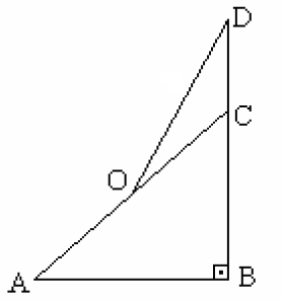
23. * If a projectile is fired with velocity v at an angle θ, then its vertical displacement y as a function of its horizontal displacement x is modeled by the parabola $y=-\frac{5}{v^2\cos^2\theta}x^2+(\tan\theta)x$. Find an expression for the range R (the maximum horizontal displacement) of this projectile.

(a) $\frac{v^2\sin 2\theta}{10}$ (b) $\frac{v^2\sin\theta}{5}$ (c) $\frac{v^2\sin\theta\cos\theta}{10}$ (d) $\frac{v^2\cos^2\theta}{20}$ (e) $\frac{v^2\sin^2\theta}{20}$

24. When a camera flash goes off, the batteries immediately begin to recharge the flash's capacitor, which stores electric charge given by $Q(t)=Q_0\left(1-e^{-t/4}\right)$. (The maximum charge capacity is Q_0 and t is measured in seconds). How long (in seconds) does it take to recharge the capacitor to within 10% of its capacity?

(a) $-4\ln 1$ (b) $-4\ln\frac{9}{10}$ (c) $-4\ln 10$ (d) $-4\ln\frac{1}{10}$ (e) $-\ln\frac{4}{9}$

25. * ABC is an isosceles right triangle. $\overline{BC}$ is extended such that B, C, and D are collinear, and O is the midpoint of $\overline{AC}$. If $BD = AC = 2$, find OD.

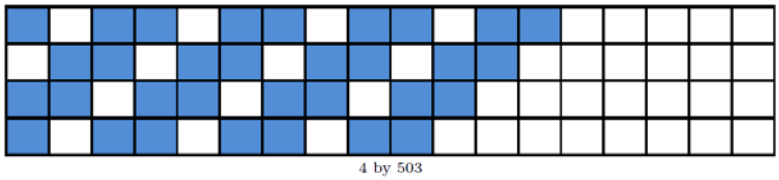


(a) $\sqrt{3-\sqrt{2}}$ (b) $\sqrt{4-2\sqrt{2}}$ (c) $\sqrt{5-\sqrt{3}}$ (d) $\sqrt{4-\sqrt{2}}$ (e) $\sqrt{5-2\sqrt{2}}$

26. A regular polygon of side length $\sqrt{2-\sqrt{3}}$ is inscribed in a unit circle. What is the number of sides of this regular polygon?

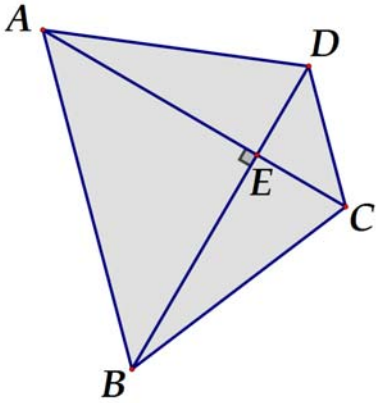
(a) 20 (b) 18 (c) 15 (d) 13 (e) 12

27. * A 4 by 503 chess board is colored in grey and white patterns as shown. How many grey squares are there of the total of 2012 squares of this chess board?



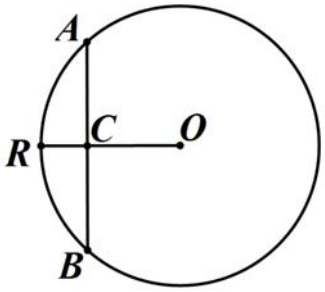
(a) 1343 (b) 1340 (c) 1341 (d) 1342 (e) 1344

28. * $ABCD$ is an isosceles trapezoid with perpendicular diagonals and base lengths $DC = 2$ and $AB = 6$. Determine the area of this trapezoid.



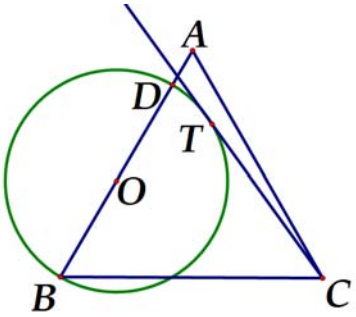
(a) 14 (b) 16 (c) 18 (d) 20 (e) 22

29. In a circle centered at O, radius $\overline{OR}$ is perpendicular to chord $\overline{AB}$ at point C. If $OC = 15$ and $CR = 2$, find AB.



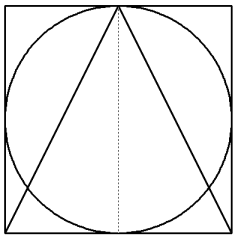
(a) 8 (b) 10 (c) 12 (d) 14 (e) 16

30. ABC is an equilateral triangle with side length 16. The point O lies on line segment $\overline{AB}$. Ray $\overrightarrow{CT}$ is tangent at the point T to the circle of radius 7 centered at O. Determine the length of the line segment $\overline{CT}$.



(a) 10 (b) 11 (c) 12 (d) 13 (e) 14

31. The Greek mathematician Archimedes liked the design below so much that he wanted it on his tombstone.



When each of the figures is revolved about the vertical line of symmetry, it generates a solid of revolution – a cylinder, a sphere, or a cone. Find the ratio of volumes $V_{CYLINDER} : V_{SPHERE} : V_{CONE}$.

(a) $3:2:1$ (b) $\sqrt{3}:\frac{\pi}{\sqrt{2}}:1$ (c) $4:\pi:2$ (d) $3:\frac{2\pi}{3}:1$ (e) $2\sqrt{\pi}:\frac{\pi}{\sqrt{2}}:\sqrt{\frac{3}{2}}$

32. Find the area of a square with the same perimeter as a regular hexagon of area $54\sqrt{3}$ square units.

(a) 9 (b) 36 (c) 81 (d) 144 (e) 225

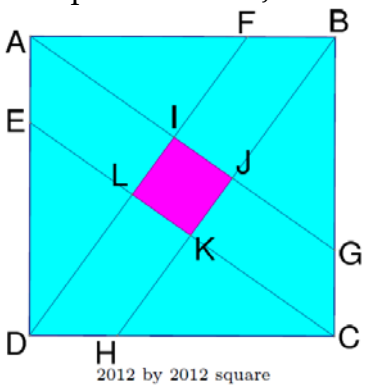
33. * In how many ways can Team USA's season of 10 soccer games result in 5 wins, 3 draws, and 2 losses in the 2014 World Cup qualification?

(a) 60 (b) 600 (c) 2520 (d) 59049 (e) 216000

34. * The Golden Ratio φ is a positive number that satisfies the quadratic equation $x^2 - x - 1 = 0$. By substitution, all positive integer powers of φ can be expressed in the form $a\varphi + b$. Determine the value of $a + b$ for φ^4.

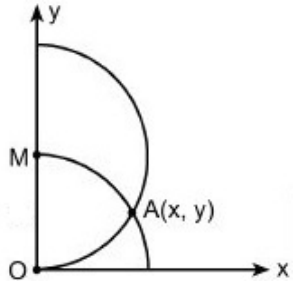
(a) 3 (b) 5 (c) 8 (d) 13 (e) 21

35. * On square *ABCD* with side length 2012, points *E*, *F*, *G*, and *H* on each side are chosen in such a way that $AF = BG = CH = DE = 1509$. Determine the ratio of the area of the square *ABCD* to the area of the quadrilateral *IJKL*.



(a) 25 (b) 24 (c) 23 (d) 22 (e) 21

36. * A semicircle of radius 2 with center $M(0,2)$ and a quarter circle of radius 2 centered at the origin intersect at a point $A(x, y)$ in the first quadrant. Find the value of x.



(a) $\frac{5}{3}$ (b) $\sqrt{2}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{3}{2}$ (e) $\sqrt{3}$

37. A locus is the set of points that satisfy a given condition. Determine an equation of the locus of complex numbers $z = x + iy$ satisfying the equation $|z + 2 - i| = 10$.

(a) $(x-1)^2 + (y-1)^2 = 100$
(b) $(x-3)^2 + (y-1)^2 = 64$
(c) $(x+2)^2 + (y-1)^2 = 100$
(d) $(x-4)^2 + (y-1)^2 = 64$
(e) $(x-4)^2 + (y-4)^2 = 100$

38. Consider the exponential function $f(x) = a^x$ where a is a positive number different from 1. For what value of a does the slope of the tangent line to the graph of f at its y-intercept equal 1?

(a) 0.5 (b) $\sqrt{2}$ (c) 2 (d) e (e) π

39. Find the coordinates of the point on the circle $x^2 + y^2 = 4$ that has the minimum distance from the line $3x + 4y - 12 = 0$.

(a) $\left(\frac{6}{5}, \frac{8}{5}\right)$ (b) $\left(\frac{8}{5}, \frac{6}{5}\right)$ (c) $\left(\frac{12}{5}, \frac{16}{5}\right)$ (d) $\left(\frac{16}{5}, \frac{12}{5}\right)$ (e) $\left(\frac{5}{6}, \frac{4}{3}\right)$

40. * If $\cos x - \sin x = \frac{1}{2}$, what is the value of $\cos 2x$ if x is in the interval $\left[0, \frac{\pi}{2}\right]$?

(a) $\frac{\sqrt{7}}{4}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $-\frac{1}{4}$ (e) -1

41. Compute $\left|\sum_{k=1}^{n} i^k\right|$ where $i=\sqrt{-1}$ and n is the largest perfect cube less than 100.

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

42. * Given that the real-valued function f satisfies the functional equation $f(x)-4f\left(\frac{1}{x}\right)=x$ for all $x>0$, find the minimum value of $\left|f(x)\right|$.

(a) $\frac{4}{15}$ (b) $\frac{7}{15}$ (c) $\frac{1}{2}$ (d) $\frac{4}{5}$ (e) $\frac{3}{5}$

43. Find the sum $\sum_{k=1}^{2012}\left(\frac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}}\right)$.

(a) $\sqrt{\frac{2012}{2013}}$ (b) $\frac{1}{\sqrt{2013}}$ (c) $\frac{\sqrt{2013}-1}{\sqrt{2013}}$ (d) $\sqrt{2013}$ (e) $\frac{\sqrt{2013}-\sqrt{2012}}{\sqrt{2013}}$

44. * If $a>1$, what is the value of the following limit?

$$\lim_{x\to 0}\frac{\sqrt[q]{1+ax}-\sqrt[q]{1-ax}}{x}$$

(a) a (b) $a+1$ (c) 2 (d) e (e) $2a$

45. Determine the value of the following limit.

$$\lim_{x\to\infty}\left(\frac{2x+5}{2x+3}\right)^{4x-1}$$

(a) 2 (b) 4 (c) e^2 (d) e^3 (e) e^4

46. What is the value of the largest possible real number δ such that $\left|\sqrt{4x+1}-3\right|<0.5$ whenever $\left|x-2\right|<\delta$?

(a) 3/8 (b) 5/8 (c) 7/8 (d) 11/16 (e) 13/16

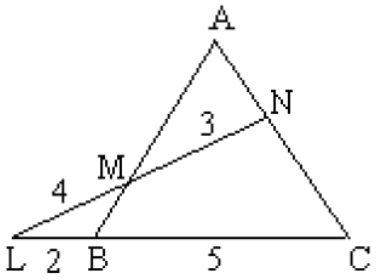
47. Let a, b, c be positive integers. How many ordered triples (a, b, c) satisfy the equation $a^2 + b^2 + c^2 = 2012$?

(a) 10 (b) 8 (c) 4 (d) 1 (e) 0

48. * Let $X = \left\{ x = \frac{n}{2} + \frac{m}{5} \middle| m, n \in \{0,1,2,\cdots,100\} \right\}$. Find the sum of the elements of X.

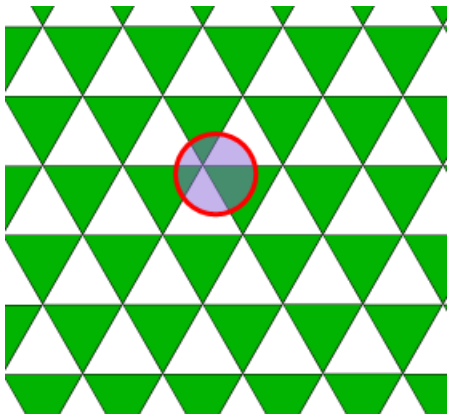
(a) 21395 (b) 22395 (c) 23395 (d) 24395 (e) 25395

49. * ABC is a triangle with $BC = 5$. A line crosses lines BC, AC and AB at points L, N and M respectively, such that $LM = 4, MN = 3, LB = 2$. Determine the ratio $\frac{NA}{NC}$.



(a) 3/7 (b) 7/15 (c) 6/17 (d) 4/15 (e) 4/21

50. * The figure below shows a tessellation of the plane with equilateral triangles of side length 1. In the experiment of tossing a coin of diameter 1, what is the probability that the coin does not overlap with any triangle vertex?



(a) $\frac{8\sqrt{3} - 4\pi}{7\sqrt{3}}$ (b) $\frac{8\sqrt{3} - 4\pi}{8\sqrt{3}}$ (c) $\frac{8\sqrt{3} - 4\pi}{9\sqrt{3}}$ (d) $\frac{8\sqrt{3} - 4\pi}{10\sqrt{3}}$ (e) $\frac{8\sqrt{3} - 4\pi}{11\sqrt{3}}$