Forty Third Annual Columbus State University
Invitational Mathematics Tournament

Sponsored by
The Columbus State University
Department of Mathematics
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The Columbus State University Mathematics faculty welcomes you to this year's tournament and to our campus. We wish you success on this test and in your future studies.

Introduction

This is a 90-minute, 50-problem, multiple-choice exam. There are five possible responses to each question. You should select the one best answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with a No.2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response. The examination will be scored on the basis of $+12$ for each correct answer, -3 for each incorrect selection, and 0 for each omitted item. Each student will be given an initial score of $+200$.

Pre-selected problems will be used as tie-breakers for individual awards. These problems, designated with an asterisk (*), in order of consideration are: 6, 24, 26, 29, 32, 34, 37, 39, 40, 41, 42, 43, 44, 46, 47, 49, and 50.

Throughout the exam, $\overline{AB}$ will denote the line segment from point A to point B and AB will denote the length of $\overline{AB}$, $\angle A$ denotes the $\angle BAC$ or $\angle CAB$ in the triangle $\triangle ABC$. Pre-drawn geometric figures are not necessarily drawn to scale.

Review and check your score sheet carefully. **Your student identification number and your school number must be encoded correctly on your score sheet.**

When you complete your test, bring your pencil, scratch paper and answer sheet to the test monitor. Leave the room after you have handed in your answer sheet. Please leave quietly so as not to disturb the other contestants. Do not congregate outside the doors by the testing area. You may keep your copy of the test. Your sponsor will have a copy of solutions to the test problems.

Do not open your test until instructed to do so!

1. If lines $2y+3x-4=0$ and $3y-ax-4=0$ intersect at a point (x_1, y_1), what is the value of $\frac{y_1}{x_1}-a$?

   $A)$ 5 $\quad B)$ 4 $\quad C)$ 2 $\quad D)$ 3 $\quad E)$ 1

   Solution: Since $2y_1+3x_1-4=0$ and $3y_1-ax_1-4=0$, we have $3y_1-ax_1=2y_1+3x_1$. Note that $x_1 \neq 0$. Hence, $\frac{y_1}{x_1}=a+3$. We have $\frac{y_1}{x_1}-a=3$. The correct answer is D.

2. A fair coin is flipped three times. What is the probability that at least one head will be thrown?

   $A)$ 0.875 $\quad B)$ 0.125 $\quad C)$ 0.25 $\quad D)$ 0.5 $\quad E)$ 0.75

   Solution: The probability to throw a head (or tails) is $\frac{1}{2}$ when the coin is flipped once. The probability that no head will be thrown is $\frac{1}{2^3}$, when a fair coin is flipped three times. Therefore, the probability that at least one head will be thrown is $1-\frac{1}{2^3}=0.875$. The correct answer is A.

3. If $i^2=-1$, find the value of i^{2017}.

   $A)$ i $\quad B)$ $-i$ $\quad C)$ 1 $\quad D)$ -1 $\quad E)$ 0

   Solution: We have

   $$i^{2017}=(i^2)^{1008}i=(-1)^{1008}i=i.$$

   The correct answer is A.

4. If $(0.67)^H=0.5$, what is the value of $2^4(0.67)^{3H}$?

   $A)$ 3 $\quad B)$ 2 $\quad C)$ 3 $\quad D)$ 4 $\quad E)$ 5

   Solution: Since

   $$2^4(0.67)^{3H}=16(0.5)^3=16\cdot 0.125=2,$$

   the correct answer is B.

5. How many real numbers are solutions of the following equation?

   $$x-\frac{3}{(x-2)^4}=2-\frac{3}{(x-2)^4}$$

   $A)$ 0 $\quad B)$ 1 $\quad C)$ 2 $\quad D)$ 3 $\quad E)$ 4

   Solution: Inspecting the equation, there is no solution. The correct answer is A.

6. * How many different nonnegative real numbers satisfy the following equation?

$$(x^2+4x-2)^2=(5x^2+2)^2$$

$A)$ 0 $B)$ 1 $C)$ 2 $D)$ 3 $E)$ 4

Solution: We have $a^2=b^2$ if and only if $a=b$ or $a=-b$. The equation $x^2+4x-2=5x^2+2$ has no real solution. The equation $x^2+4x-2=-5x^2-2$ has two distinct real solutions $x=0$ and $x=-\frac{2}{3}$. Therefore there is only one nonnegative real solution of our original equation. The correct answer is B.

7. How many real numbers x satisfy equation $(x^2-x+1)^{(x-1)}=1$?

$A)$ 0 $B)$ 1 $C)$ 2 $D)$ 3 $E)$ 4

Solution: We need to consider two cases:

Case 1: $x^2-x+1=1$. We have $x=0$ or $x=1$.

Case 2: $x^2-x+1\neq 0$ and $x-1=0$. We have $x=1$.

The correct answer is C.

8. A certain two digit number is equal to twice the sum of its digits. What is the product of its digits?

$A)$ 4 $B)$ 6 $C)$ 7 $D)$ 5 $E)$ 8

Solution: Suppose a is the tens' place value and b is the ones' place value of the number. Then $10a+b=2(a+b)$, so $8a=b$. That is $a=1$ and $b=8$. The correct answer is E.

9. Find the area of the region bounded by the $x-$axis, $y-$axis, and $\frac{x}{3}-\frac{y}{4}=1$.

$A)$ 10 $B)$ 9 $C)$ 3 $D)$ 7 $E)$ 6

Solution: The bounded region is a triangle with vertexes $(0,0)$, $(3,0)$, and $(0,-4)$. Its area is equal to $3\cdot 4/2=6$. The correct answer is E.

10. Assume a and b are constants such that the following equality

$$\frac{a}{x-1}+\frac{b}{x-2}=\frac{2017}{(x-1)(x-2)}$$

is true for any real number x except 1 and 2. What is the value of $a+b$?

$A)$ 4 $B)$ 3 $C)$ 2 $D)$ 1 $E)$ 0

Solution: Since

$$\frac{a}{x-1}+\frac{b}{x-2}=\frac{a(x-2)+b(x-1)}{(x-1)(x-2)}=\frac{(a+b)x-(2a+b)}{(x-1)(x-2)},$$

we have $a(x-2)+b(x-1)=2017$ for any real number if and only if

$$a+b=0,$$
$$2a+b=-2017.$$

The correct answer is E.

11. Suppose that a and b are real numbers such that $1+\sqrt{3}i$ is a root of the quadratic equation $\frac{x^2}{2}+ax+b=0$. Find the value of b.

$A)\,0$ $B)\,1$ $C)\,2$ $D)\,3$ $E)\,4$

Solution: Note that $1-\sqrt{3}i$ is the second root of the equation. Since

$$(1-\sqrt{3}i)(1+\sqrt{3}i)=4=2b,$$

we have $b=2$. The correct answer is C.

12. If $9^{1-\frac{3}{2}x}=\pi$, find the value of $6\cdot\pi(27)^{x-1}$.

$A)\,1$ $B)\,2$ $C)\,3$ $D)\,4$ $E)\,5$

Solution: From $9^{1-\frac{3}{2}x}=\pi$, we have $3^{3x}=\frac{9}{\pi}$. Consequently

$$6\cdot\pi\cdot(27)^{x-1}=6\cdot\pi\cdot3^{3x-3}=\frac{6\pi}{27}\cdot3^{3x}=\frac{6\pi}{27}\cdot\frac{9}{\pi}=2.$$

The correct answer is B.

13. Let x, y be real numbers such that

$$\begin{cases} xy-x=25, \\ xy+y=36. \end{cases}$$

What is the value of $y-x$?

$A)\,1$ $B)\,2$ $C)\,0$ $D)\,-2$ $E)\,-1$

Solution: By subtracting the given equations, we have $x+y=11$, so $y=11-x$. The first equation becomes $x(10-x)=25$. That is, $(x-5)^2=0$. So $x=5$ and $y=6$. The correct answer is A.

14. What is the product of the roots of the following equation?

$$(x+1)(x-2)+(x+2)(x-3)+(x+3)(x-1)+(x+3)(x+5)=0.$$

$A)$ 2 $B)$ 1 $C)$ 0 $D)$ 3 $E)$ 5

Solution: Expanding the given equation, we have

$(x+1)(x-2)+(x+2)(x-3)+(x+3)(x-1)+(x+3)(x+5)=4(x^2+2x+1)=0.$

So $x_1=x_2=-1$. The correct answer is B.

15. The quadratic function $f(x)=sx^2+tx+r$ satisfies the equation $f(x)-f(x-1)=4x+1$ for all x. What is the value of $t-s$?

$A)$ 1 $B)$ 2 $C)$ 3 $D)$ 4 $E)$ 5

Solution: Since

$$f(x)-f(x-1)=sx^2+tx-s(x-1)^2-t(x-1)=2sx+t-s=4x+1.$$

Expanding the coefficients of x gives $s=2$ and $t=3$. The correct answer is A.

16. If $\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{2017}{a+b}$, find the value of $\dfrac{b}{a}+\dfrac{a}{b}$.

$A)$ 2014 $B)$ 2015 $C)$ 2016 $D)$ 2017 $E)$ 2018

Solution: Multiplying both sides of equation $\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{2017}{a+b}$ by $a+b$ gives $\dfrac{a+b}{a}+\dfrac{a+b}{b}=2017$ and $1+\dfrac{b}{a}+\dfrac{a}{b}+1=2017$. Thus $\dfrac{b}{a}+\dfrac{a}{b}=2015$. The correct answer is B.

17. Find the length of the **minor** axis(the shortest diameter of an ellipse) of the ellipse $x^2+5y^2=5$.

$A)$ 1 $B)$ 2 $C)$ 3 $D)$ 4 $E)$ 5

Solution: Note that

$$x^2+5y^2=5 \Rightarrow \frac{x^2}{5}+y^2=1.$$

The correct answer is B.

18. Given numbers k, l and m such that

$$x^3+kx^2-x-2=(lx-1)(x^2+mx+2)$$

is an identity in x, what is the value of $m-k$?

$A)$ 3 $\quad$ $B)$ 4 $\quad$ $C)$ 0 $\quad$ $D)$ 1 $\quad$ $E)$ 2

Solution: Expanding gives

$$x^3 + kx^2 - x - 2 = lx^3 + (lm-1)x^2 + (2l-m)x - 2.$$

Therefore, we have three equalities $l = 1$, $lm = k+1$ and $2l - m = -1$. It is easy to see $m = 3$ and $k = 2$. The correct answer is D.

19. Let $\frac{a+2}{a+1} = pa + q$ and $2a^2 - 3a - 8 = 0$. Find the value of $p + q$.

$A)$ 0 $\quad$ $B)$ 1 $\quad$ $C)$ 2 $\quad$ $D)$ 3 $\quad$ $E)$ 4

Solution: The equation $\frac{a+2}{a+1} = pa + q$ is equivalent to

$$pa^2 + (p+q-1)a + q - 2 = 0,$$

which has the same roots as the equation $2a^2 - 3a - 8 = 0$. Therefore,

$$\frac{p}{2} = \frac{p+q-1}{-3} = \frac{q-2}{-8}.$$

We obtain

$$\begin{cases} -3p = 2(p+q-1) \\ -8(p+q-1) = -3(q-2), \end{cases}$$

so $p = \frac{2}{3}$ and $q = -\frac{2}{3}$. The correct answer is A.

20. Let F be a function from positive real numbers to positive real numbers such that $F(2) = a$. If $[F(xy)]^2 = x[F(y)]^2$ for all positive real numbers x and y, find the value of $F(50)$.

$A)$ a $\quad$ $B)$ $2a$ $\quad$ $C)$ $3a$ $\quad$ $D)$ $4a$ $\quad$ $E)$ $5a$

Solution: From $[F(xy)]^2 = x[F(y)]^2$, we have $F(xy) = \sqrt{x}F(y)$. Letting $x = 25$ and $y = 2$, we have $F(50) = \sqrt{25}F(2) = 5a$. The correct answer is E.

21. If $\log_a x = 4$, what is the value of $\log_{\frac{1}{a}} x$?

$A)$ $-\dfrac{1}{4}$ $\quad$ $B)$ -4 $\quad$ $C)$ 1 $\quad$ $D)$ 4 $\quad$ $E)$ $\dfrac{1}{4}$

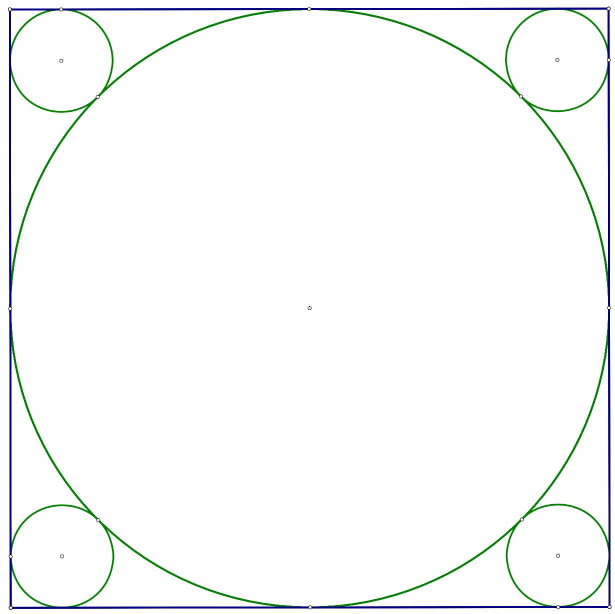
Solution: Since $\log_{\frac{1}{a}} x = -\log_a x = -4$, the correct answer is B.

22. A bag contains 10 chips that are either red or blue. It is known that the probability of selecting (without replacement) a red chip and then a blue chip (in that order) is $\frac{4}{15}$, and that the probability of selecting two blue chips is $\frac{1}{3}$. How many red chips are in the bag?

$A)$ 3 $\quad$ $B)$ 4 $\quad$ $C)$ 5 $\quad$ $D)$ 6 $\quad$ $E)$ 2

Solution: If x, y denote the number of red and blue chips, respectively, we have the equations: $x+y = 10$, $\frac{x}{10}\frac{y}{9} = \frac{4}{15}$, and $\frac{y}{10}\frac{y-1}{9} = \frac{1}{3}$. Solving these equations leads to $x = 4$ and $y = 6$. The correct answer is B.

23. In the accompanying figure we have a square and a large circle is inscribed in the square. The smaller circles are tangent to the large circle and to the sides of the square. If the sides of the square are equal to 2 cm, then the radii of the smaller circles are equal to $a - b\sqrt{2}$ cm, for some natural numbers a and b. What is $a - b$?



$A)$ 1 $B)$ 2 $C)$ 3 $D)$ 4 $E)$ 0

Solution: If the radius of the small circles is r, then we have $2\sqrt{2} = 2 + 2r + 2\sqrt{2}r$. Therefore we have $r = (\sqrt{2} - 1)^2 = 3 - 2\sqrt{2}$. The correct answer is A.

24. * What is the coefficient of x^4y^2 in the expansion of $(x + 2y)^6$?

$A)$ 20 $B)$ 30 $C)$ 40 $D)$ 50 $E)$ 60

Solution: By the Binomial Theorem, the coefficient of x^4y^2 is

$$\binom{6}{4} 2^2 = \frac{6!}{4!2!} 4 = 6 \cdot 5 \cdot 2 = 60.$$

The correct answer is E.

25. The equation $r = 6\cos\theta$ is a circle written in polar form. What is the radius of the circle?

$A)$ 6 $B)$ 5 $C)$ 4 $D)$ 3 $E)$ 2

Solution: From $r = 6\cos\theta$, we have $r^2 = 6r\cos\theta$. Since $x = r\cos\theta$ and $y = r\sin\theta$, then $(x-3)^2 + y^2 = 9$. The correct answer is D.

26. * Compute the value of $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{300}$, where $i^2=-1$.

$A)\ -1$ $\quad$ $B)\ 1$ $\quad$ $C)\ 2$ $\quad$ $D)\ \sqrt{3}$ $\quad$ $E)\ -\sqrt{3}$

Solution: Since

$$\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{300}=\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{300}=(\cos\pi+i\sin\pi)^{100}=1,$$

the correct answer is B.

27. Let a and b be complex numbers such that $\frac{a}{b}+\frac{b}{a}=\sqrt{3}$. What is the value of a^6+b^6 ?

$A)\ 1$ $\quad$ $B)\ 2$ $\quad$ $C)\ 3$ $\quad$ $D)\ 4$ $\quad$ $E)\ 0$

Solution: From $\frac{a}{b}+\frac{b}{a}=\sqrt{3}$, we have $a^2+b^2=\sqrt{3}ab$ and $a^4+b^4-a^2b^2=0$.

$$a^6+b^6=(a^2+b^2)(a^4+b^4-a^2b^2)=0.$$

The correct answer is E.

28. If $\log_x 64=3$, find the value of $\log_2\left(x^3\right)$.

$A)\ 5$ $\quad$ $B)\ 4$ $\quad$ $C)\ 6$ $\quad$ $D)\ 2$ $\quad$ $E)\ 3$

Solution: If $\log_x 64=3$, then $x=4$. Therefore, $\log_2\left(x^3\right)=\log_2(64)=6$. The correct answer is C.

29. * For each real number m the parabola $y=(m^2+2)x^2+(m-1)^2x-2m+2$ passes through the same point (a,b). What is the value of $(a+b)^2$?

$A)\ 0$ $\quad$ $B)\ 1$ $\quad$ $C)\ 2$ $\quad$ $D)\ 3$ $\quad$ $E)\ 4$

Solution: Substitute a,b in the above expression and arrange according to the powers of m.

$$m^2(a^2+a)-2m(a+1)+2a^2+a+2-b=0$$

for each real number m. This means that $a^2+a=0$, $a+1=0$ and $2a^2+a+2-b=0$. This implies $a=-1$ and $b=3$. The correct answer is E.

30. Find the product of the solutions of the equation

$$\log_3 x-4\log_x 3-3=0.$$

$A)\ 12$ $\quad$ $B)\ 9$ $\quad$ $C)\ 16$ $\quad$ $D)\ 27$ $\quad$ $E)\ 36$

Solution: The equation is equivalent to

$$(\log_3 x - 4)(\log_3 x + 1) = 0.$$

This implies $x = 3^4$ and $x = 3^{-1}$, so the product of the solutions is 27. The correct answer is D.

31. Find the interval on which the function $f(x) = \log_{\frac{1}{2}}(-x^2 - 2x + 3)$ is increasing.

$A)$ $(-\infty, +\infty)$ $B)$ $(-\infty, -3)$ $C)$ $\big(-3, -1\big]$ $D)$ $\big(-1, 1\big)$ $E)$ $\big[-1, 1\big]$

Solution: Denote $g(x) = -x^2 - 2x + 3$. From $g(x) > 0$, we have $-3 < x < 1$. The function $g(x)$ is decreasing on the interval $(-1, 1)$. Note that the function $f(x)$ is a decreasing function. Hence, $f(x)$ is increasing on the interval $(-1, 1)$. The correct answer is D.

32. * Find the number of nonnegative integers n such that $\dfrac{n^2}{n+6}$ is an integer.

$A)$ 1 $B)$ 2 $C)$ 3 $D)$ 4 $E)$ 5

Solution: Note that $\dfrac{n^2}{n+6} = n - 6 + \dfrac{36}{n+6}$, so the problem is equivalent to finding all nonnegative integers n such that $n+6$ divides 36. This implies that $n \in \{0, 3, 6, 12, 30\}$ The correct answer is E.

33. What is the value of the product $\left(1 + \dfrac{1}{2}\right)\left(1 + \dfrac{1}{3}\right)\cdots\left(1 + \dfrac{1}{2017}\right)$?

$A)$ $\dfrac{2017}{2}$ $B)$ 2017 $C)$ $\dfrac{2018}{3}$ $D)$ 1009 $E)$ 1010

Solution: $\left(1 + \dfrac{1}{2}\right)\left(1 + \dfrac{1}{3}\right)\cdots\left(1 + \dfrac{1}{2017}\right) = \dfrac{3}{2}\dfrac{4}{3}\dfrac{5}{4}\cdots\dfrac{2018}{2017} = \dfrac{2018}{2}$. The correct answer is D.

34. * Solve the equation $x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}$.

$A)$ 1 $B)$ 2 $C)$ 0 $D)$ -2 $E)$ -1

Solution: Squared the given equation, we have

$$x^2 = 2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}} = 2 + x.$$

Clearly, $x = 2$ is the only solution. The correct answer is B.

35. A point is chosen at random on a line segment of length 1 dividing it into two segments. Find the probability that the ratio of the shorter to the longer segment is less than $\frac{1}{4}$.

$A)\ \dfrac{2}{5}$ $B)\ \dfrac{1}{5}$ $C)\ \dfrac{3}{5}$ $D)\ \dfrac{1}{3}$ $E)\ \dfrac{2}{3}$

Solution: We will take the interval $[0, 1]$ as the segment of length 1. Let us denote by x the selected point on the segment. The ratio of the shorter to the longer segment is $\frac{x}{1-x}$ or $\frac{1-x}{x}$ depending on the value of x. Thus we need to solve $\frac{x}{1-x} < \frac{1}{4}$ and $\frac{1-x}{x} < \frac{1}{4}$. We obtain $x < \frac{1}{5}$ and $\frac{4}{5} < x$. Since all points are equally likely to be selected, the probability that the ratio of the shorter to the longer segment is less than $\frac{1}{4}$ is $\frac{1}{5} + \frac{1}{5} = \frac{2}{5}$. The correct answer is A.

36. Let $S = \cos 1^\circ + \cos 3^\circ + \cos 5^\circ + \cdots + \cos 29^\circ$. What is the value of $S \sin 1^\circ$?

$A)\ \dfrac{5}{4}$ $B)\ 1$ $C)\ \dfrac{3}{4}$ $D)\ \dfrac{1}{4}$ $E)\ \dfrac{1}{2}$

Solution:

$$\begin{aligned}\sin 1^\circ S &= (2\sin 1^\circ \cos 1^\circ + 2\sin 1^\circ \cos 3^\circ + \cdots + 2\sin 1^\circ \cos 29^\circ)/2\\ &= (\sin 2^\circ - \sin 0^\circ + \sin 4^\circ - \sin 2^\circ + \sin 6^\circ - \sin 4^\circ + \cdots + \sin 30^\circ - \sin 28^\circ)/2\\ &= \sin 30^\circ/2 = \frac{1}{4}.\end{aligned}$$

The correct answer is D.

37. * What is the 2017^{th} term of the sequence

$$1,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4,\ 4,\ 4, \ldots ?$$

$A)\ 62$ $B)\ 63$ $C)\ 64$ $D)\ 65$ $E)\ 66$

Solution: Let n be the value of the 2017^{th} term. We have

$$1 + 2 + 3 + \cdots + (n-1) < 2017 \leq 1 + 2 + 3 + \cdots + n.$$

This is because for each positive integer k there are exactly k terms of the sequence equal to k. The above inequalities imply $n = 64$. The correct answer is C.

38. Let $\sin\alpha\cos\alpha = \dfrac{1}{8}$ and $\pi/4 < \alpha < \pi/2$. Find the value of $\cos\alpha - \sin\alpha$.

$A)\ \dfrac{\sqrt{3}}{2}$ $B)\ -\dfrac{\sqrt{3}}{2}$ $C)\ \dfrac{3}{4}$ $D)\ -\dfrac{3}{4}$ $E)\ \dfrac{1}{2}$

Solution: Using

$$(\sin\alpha + \cos\alpha)^2 = 1 + 2\sin\alpha\cos\alpha = 1 + 2\cdot\left(\frac{1}{8}\right) = \frac{5}{4},$$

we have $\sin\alpha + \cos\alpha = \dfrac{\sqrt{5}}{2}$ and $\sin\alpha\cos\alpha = \dfrac{1}{8}$. This indicates that $\sin\alpha$ and $\cos\alpha$ are the two roots of the equation $x^2 - \dfrac{\sqrt{5}}{2}x + \dfrac{1}{8} = 0$. By the quadratic formula, the absolute value of the difference of the two roots is

$$|\sin\alpha - \cos\alpha| = \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - 4(1)\cdot\left(\frac{1}{8}\right)} = \frac{\sqrt{3}}{2}.$$

Since $\sin\alpha > \cos\alpha$ for $\pi/4 < \alpha < \pi/2$, the answer is B.

39. * Let θ be a real number such that $\sin\theta + \csc\theta = 2$. Find the value of $\sin^3\theta + \csc^3\theta$.

$A)\ 2\sin\theta$ $\quad B)\ -2\sin\theta$ $\quad C)\ 2\cos\theta$ $\quad D)\ -2\cos\theta$ $\quad E)\ 2$

Solution: Using the binomial formula we get

$$\begin{aligned}(\sin\theta + \csc\theta)^3 &= \sin^3\theta + 3\sin^2\theta\csc\theta + 3\sin\theta\csc^2\theta + \csc^3\theta \\ &= \sin^3\theta + 3(\sin\theta + \csc\theta) + \csc^3\theta \\ &= \sin^3\theta + 6 + \csc^3\theta = 8.\end{aligned}$$

The correct answer is E.

40. * Let $P(x, y)$ be any point on the circle $r = 2\cos\theta - 2\sin\theta$. Find the minimum value of $x^2 + y^2$.

$A)\ 2$ $\quad B)\ 1$ $\quad C)\ 0$ $\quad D)\ 3$ $\quad E)\ 4$

Solution: Since $r^2 = 2r\cos\theta - 2r\sin\theta$, the circle can be rewritten as

$$(x-1)^2 + (y+1)^2 = 2.$$

Let $x = 1 + \sqrt{2}\sin t$, $y = -1 + \sqrt{2}\cos t$. We have

$$x^2 + y^2 = 1 + 2\sqrt{2}\sin t + 2\sin^2 t + 1 - 2\sqrt{2}\cos t + 2\cos^2 t = 4 + 4\sin\left(t - \frac{\pi}{4}\right).$$

The correct answer is C.

41. * Three balls are randomly drawn, without replacement, from a bowl containing 6 yellow and 5 blue balls. Find the probability that one of the balls is yellow and the other two are blue.

$A)\ \dfrac{4}{11}$ $\quad B)\ \dfrac{3}{11}$ $\quad C)\ \dfrac{5}{11}$ $\quad D)\ \dfrac{6}{11}$ $\quad E)\ \dfrac{7}{11}$

Solution: Denote the probability for an event to happen $P(\text{event})$. With this notation, the desired probability, $P(\text{drawing 1 yellow and 2 blue})$, is equal to

$$\begin{aligned}&P(\text{drawing first yellow, second and third blue}) \\ &+ P(\text{drawing first blue, second yellow, third blue}) \\ &+ P(\text{drawing first and second blue, third yellow}) \\ &= \frac{6}{11}\frac{5}{10}\frac{4}{9} + \frac{5}{11}\frac{6}{10}\frac{4}{9} + \frac{5}{11}\frac{4}{10}\frac{6}{9} = \frac{4}{11}.\end{aligned}$$

The correct answer is A.

42. * Let a, b, c, and d be positive real numbers such that $ab = cd = 1$. Find the smallest possible value of the expression

$$S = \frac{(a+1)(b+1)(c+3)(d+3)}{16}.$$

A) 1 *B*) 2 *C*) 3 *D*) 4 *E*) 5

Solution: Expanding, we have

$$16S = (ab + a + b + 1)(cd + 3(c+d) + 9) = \left(2 + a + \frac{1}{a}\right)\left(10 + 3\left(c + \frac{1}{c}\right)\right)$$
$$\geq \left(2 + 2\sqrt{a \cdot \frac{1}{a}}\right)\left(10 + 3\left(2\sqrt{c \cdot \frac{1}{c}}\right)\right) = 64.$$

Hence, $S \geq 4$. With $a = c = 1$, $S = 4$. The correct answer is D.

43. * Suppose that $x_1, x_2, \cdots, x_{2017}$ are the 2017 roots of the polynomial

$$P(x) = x^{2017} - 2017^{2016}x - 2.$$

Find the average of $x_1^{2017}, x_2^{2017}, \cdots, x_{2017}^{2017}$.

A) -2 *B*) -1 *C*) 0 *D*) 2 *E*) 3

Solution: Since $P(x_i) = x_i^{2017} - 2017^{2016}x_i - 2 = 0$ for all $i = 1, 2, \cdots, 2017$, we have

$$0 = \sum_{i=1}^{2017}(x_i^{2017} - 2017^{2016}x_i - 2)$$
$$= \sum_{i=1}^{2017} x_i^{2017} - 2017^{2016}\sum_{i=1}^{2017} x_i - 2017 \cdot 2 = \sum_{i=1}^{2017} x_i^{2017} - 2017 \cdot 2.$$

$\sum_{i=1}^{2017} x_i$ is equal to the coefficient of x^{2016} in the given polynomial, which is zero. The correct answer is D.

44. * Let a, b, c, and d be positive real numbers such that $\dfrac{a+b}{c+d} = \dfrac{2a+b}{2c+d} = \dfrac{3a+b}{3c+d} = 3$. What is the value of $\dfrac{2017a+b}{2017c+d}$?

A) 2014 *B*) 2015 *C*) 1 *D*) 2 *E*) 3

Solution: Use cross multiplication of $\dfrac{a+b}{c+d} = \dfrac{2a+b}{2c+d}$ to get $ad = bc$. This implies $\dfrac{a}{c} = \dfrac{b}{d} = k$, where k is a real number. Substituting $a = kc$ and $b = dk$ into $\dfrac{3a+b}{3c+d} = 3$, we have $k = 3$. Hence, $\dfrac{2017a+b}{2017c+d} = \dfrac{2017 \cdot 3c + 3d}{2017c+d} = \dfrac{3(2017c+d)}{2017c+d} = 3$. The correct answer is E.

45. If $A = 3^{2017} - 3^{2016} + 3^{2015} - 3^{2014}$, what are the tens and ones digits of A?

$A)$ 10 $B)$ 20 $C)$ 40 $D)$ 60 $E)$ 80

Solution:

$$\begin{aligned} A &= 3^{2017} - 3^{2016} + 3^{2015} - 3^{2014} = 3^{2014}(27 - 9 + 3 - 1) \\ &= 20 \cdot 3^{2014} = 20 \cdot 9^{1007} = 20(10 - 1)^{1007} \end{aligned}$$

Considering the binomial expansion, the tens and ones digits of A are the same as the tens and ones digits of $20 \cdot \left[\binom{1007}{1}10 - 1\right]$, so we get 80. The correct answer is E.

46. * Let a, b, and c be positive numbers such that $a + b + c = 9$ and

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = \frac{1}{3}.$$

What is the value of $\dfrac{a}{b+c} + \dfrac{b}{c+a} + \dfrac{c}{a+b}$?

$A)$ 1 $B)$ 2 $C)$ 0 $D)$ 4 $E)$ 3

Solution: Clearly,

$$\begin{aligned} \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} &= \frac{9-(b+c)}{b+c} + \frac{9-(c+a)}{c+a} + \frac{9-(a+b)}{a+b} \\ &= \frac{9}{b+c} + \frac{9}{c+a} + \frac{9}{a+b} - 3 = 0. \end{aligned}$$

The correct answer is C.

47. * If positive integers x and y satisfy the equation $2017 + 2016i = (48 + 16i)^2 + (y + xi)^2$, what is $x - y$?

$A)$ 1 $B)$ 2 $C)$ 3 $D)$ 4 $E)$ 5

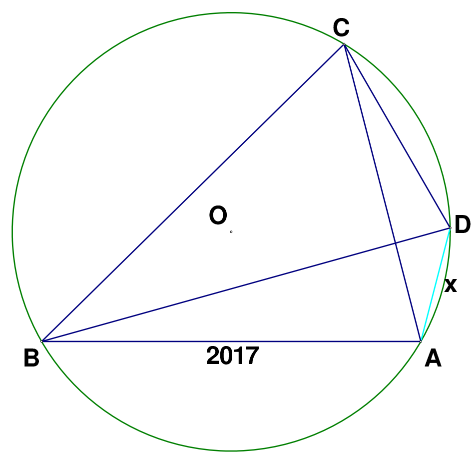
Solution: Equating the real and imaginary parts, we obtain $x^2 - y^2 = 31$ and $2xy = 480$. Note that 31 is a prime number and $(x + y)(x - y) = 31$. We get $x = 16$ and $y = 15$. The correct answer is A.

48. Two points are chosen at random on a square (uniform distribution on each side). The probability that the distance between them is more than or equal to the side of the square is equal to $\frac{m-\pi}{n}$ for two natural numbers m and n. What is $n - m$?

$A)$ 1 $B)$ 2 $C)$ 3 $D)$ 4 $E)$ 5

Solution: We can fix one point on a side. The other point can be on the opposite side or on an adjacent side. The probability to be on the opposite side is $1/4$ and the probability to be on an adjacent side is $1/2$. If the second point is on the opposite side, the distance is always more than or equal to the side of the square. So, we only need to calculate the probability in case the second point is on an adjacent side. This is easily seen to be equal to $1 - \int_0^1 \sqrt{1 - x^2}dx = 1 - \frac{\pi}{4}$, where x denotes the location of the first point. Then the probability we are looking for is $1/4 + (1/2)(1 - \pi/4) = \frac{6-\pi}{8}$. The correct answer is B.

49. * In the accompanying figure ABCD is a cyclic quadrilateral (inscribed in a circle) in which $AB = 2017$ cm, $BD = BC = 2255$ cm, and $AC = 1632$ cm. The length of the chord $x = AD$ (in centimeters) is a number of three digits (in base 10). What is the difference between the first and the second digit (the hundreds digit minus the tens digit) of x?.



$A)$ 1 $B)$ 2 $C)$ 3 $D)$ 4 $E)$ 5

Solution: Use the Law of Cosines $AB^2 = BC^2 + AC^2 - 2AC \cdot BC \cdot \cos\angle BCA$, we have

$$\cos\angle BCA = \frac{2255^2 + 1632^2 - 2017^2}{2 \cdot 2255 \cdot 1632} = \frac{1}{2}.$$

So the angle $\angle BCA = 60°$. Using the Law of Cosines again in the triangle ABD and $\angle BCA = 60° = \angle BDA$, we have

$$\begin{aligned} AB^2 &= BD^2 + x^2 - 2 \cdot BD \cdot x \cdot \cos\angle BDA \\ &\Rightarrow x^2 - 2255x + 2255^2 - 2017^2 = 0 \\ &\Rightarrow x^2 - 2255x + 1016736 = 0. \end{aligned}$$

It is easy for us to find $x = 623$. The correct answer is D.

50. * The number 1729 is the Hardy-Ramanujan number, i.e., the smallest integer which is the sum of two cubes in two different ways ($1729 = 1^3 + 12^3 = 9^3 + 10^3$). This allows us to write 2017 as a sum of five different positive cubes (of natural numbers) in two different ways:

$$2017 = 1^3 + 12^3 + a^3 + b^3 + c^3 = 9^3 + 10^3 + a^3 + b^3 + c^3.$$

If $a < b < c$, find $(a - b + c)/2$.

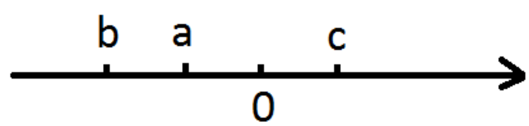
$A)$ 0 $B)$ 1 $C)$ 2 $D)$ 3 $E)$ 4

Solution: The equation reduces to $a^3 + b^3 + c^3 = 288 = 8(36)$. We know that $1^3 + 2^3 + 3^3 + ... + n^3 = (n(n+1)/2)^2$ which gives the solution $a = 2$, $b = 2 \cdot 2$ and $c = 2 \cdot 3$ right away. It is not hard to argue that this is unique. The correct answer is C.

Math Tournament 2017 - Ciphering

Round 1

1. Let a, b and c be points equally spaced in the number line as shown below.



Simplify the expression

$$\sqrt{a^2} - |a+b| + \sqrt{(c-a)^2} + |b+c|.$$

Answer : $-a$.

It is easy to see $c > 0 > a > b$, therefore

$$\sqrt{a^2} - |a+b| + \sqrt{(c-a)^2} + |b+c| = -a - (-(a+b)) + c - a - (b+c).$$

2. If $a = \sqrt{2}+1$, what is the value of $1 + \dfrac{1}{2+\frac{1}{a}}$?

Answer: $\sqrt{2}$.

$$1 + \frac{1}{2+\frac{1}{a}} = 1 + \frac{1}{2+\frac{1}{\sqrt{2}+1}} = 1 + \frac{1}{2+\sqrt{2}-1} = 1 + \frac{1}{\sqrt{2}+1} = 1 + \sqrt{2} - 1 = \sqrt{2}.$$

3. One third of the number 27^{81} can be written in the form 9^A where A is a three digit number. Find the value of A.

Answer : 121.

We have

$$\frac{1}{3} \cdot 27^{81} = \frac{1}{3} \cdot 3^{243} = 3^{242} = 9^{121}.$$

4. If $2x^2 + y^2 - 2xy - 4x + 4 = 0$, what is the value of $x + y$?

   Answer : 4.

   Rearranging the original equation, we have

$$\begin{aligned} 2x^2 + y^2 - 2xy - 4x + 4 &= x^2 + y^2 - 2xy + x^2 - 4x + 4 \\ &= (x-y)^2 + (x-2)^2 = 0. \end{aligned}$$

   Therefore, $x = y = 2$.

5. Let a and b be real numbers such that $a \neq b$, $a^2 + 3a = 2$, and $b^2 + 3b = 2$. Find the value of $(1 + a)(1 + b)$.

   Answer : -4.
   It is easy to see a and b are the roots of $x^2 + 3x - 2 = 0$. Therefore, $a + b = -3$, $ab = -2$, and

$$(1 + a)(1 + b) = 1 + a + b + ab = 1 - 3 - 2 = -4.$$

6. Evaluate $\log_2 2\sqrt{4\sqrt{2}} - \frac{1}{4}$.

   Answer : 2.

   We have

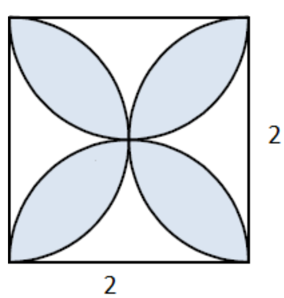
$$\log_2 2\sqrt{4\sqrt{2}} - \frac{1}{4} = \log_2 4\sqrt{2^{\frac{1}{2}}} - \frac{1}{4} = \log_2 2^{\frac{9}{4}} - \frac{1}{4} = \frac{9}{4} - \frac{1}{4} = 2.$$

7. The sum of two prime numbers is 99. What is the product of these two prime numbers?

   Answer : 194.

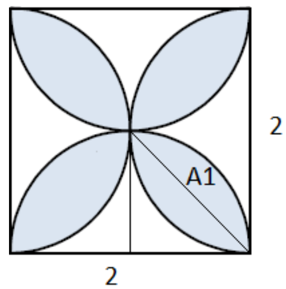
   Suppose a and b are prime numbers such that $a + b = 99$, which means either a or b will be an even number. There is only one even prime number, 2. The answer is 194.

8. The figure shows a square and four semicircles generated with each side of the square as a diameter. If the side length of the square is 2, find the area of the shaded region.



   Answer : $2\pi - 4$.

The area of A_1 can be obtained by deducting the area of the triangle from the area of the quarter circle with radius 1. We have $A_1 = \frac{\pi}{4} - \frac{1}{2}$. All shaded area$= 8 \cdot A_1 = 2\pi - 4$.



Round 2

1. How many digits are in the base-ten numeral $16^{11} \cdot 5^{40}$?

   Answer : 42

   We have
$$16^{11} \cdot 5^{40} = 2^{44} \cdot 5^{40} = 2^4 \cdot 10^{40} = 16 \cdot 10^{40}.$$
   There are 42 digits.

2. Amber, Ben, and Cathy shared a pizza. Amber ate $\frac{1}{5}$ of the pizza, Ben ate one-half as much as Cathy did. Find how much of the pizza Ben ate.

   Answer : $\frac{4}{15}$.

   Since Amber ate $\frac{1}{5}$, the remaining $\frac{4}{5}$ is shared by Ben and Cathy. Assume Cathy ate x portion of the pizza. Then Ben should ate $\frac{1}{2}x$. Therefore $x + \frac{1}{2}x = \frac{4}{5}$. Solving for x gives $x = \frac{8}{15}$. Ben ate $\frac{1}{2}x = \frac{4}{15}$ of the pizza.

3. Let a and b be real numbers such that $a + b = 1$ and $a^2 + b^2 = 4$. What is the value of $\frac{b^2}{2+a} + \frac{a^2}{2+b}$?

   Answer : 3.

   Since $b^2 = 4 - a^2$ and $a^2 = 4 - b^2$, we can simplify
$$\frac{b^2}{2+a} + \frac{a^2}{2+b} = \frac{4-a^2}{2+a} + \frac{4-b^4}{2+b} = 2 - b + 2 - a = 4 - (a+b) = 4 - 1 = 3.$$

4. Let a be a real number such that $a > 0$ and $|(a + 2i)(1 + i)| = 4$. Find the value of a.

   Answer : 2.

   Rewrite
$$|(a + 2i)(1 + i)| = \sqrt{a^2+4} \cdot \sqrt{1+1} = 4 \Rightarrow a^2 = 4 \Rightarrow a = \pm 2.$$

5. In a barn with chickens and rabbits, there are 7 heads and 20 legs. How many chickens are there? (A chicken has 2 legs and a rabbit has 4 legs.)

   Answer : 4.

   Let x denote the number of chicken and y denote the number of rabbits. Then $x+y = 7$ and $2x + 4y = 20$. We have $x = 4$ and $y = 3$.

6. Let $f(x) = (2 - \sin\sqrt{x})^2$. What is the maximum value of $f(x)$?

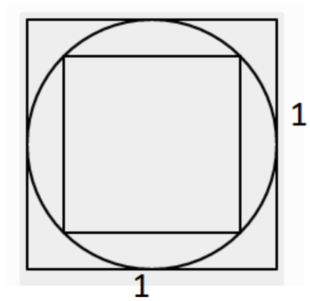
   Answer : 9.

   Since $|\sin\sqrt{x}| \le 1$, then
$$(2 - \sin\sqrt{x})^2 \le (2 + 1)^2.$$

7. Let $M = \{1, 2, 3\}$ and $N = \{2a - 1 | a \in M\}$. Find $M \bigcap N$.

   Answer : {1,3}.
   Note that $N = \{2a - 1 | a \in M\} = \{1, 3, 5\}$. The solution is $M \bigcap N = \{1, 3\}$.

8. If the area of the outer square is 1, what is the area of the small square inside?

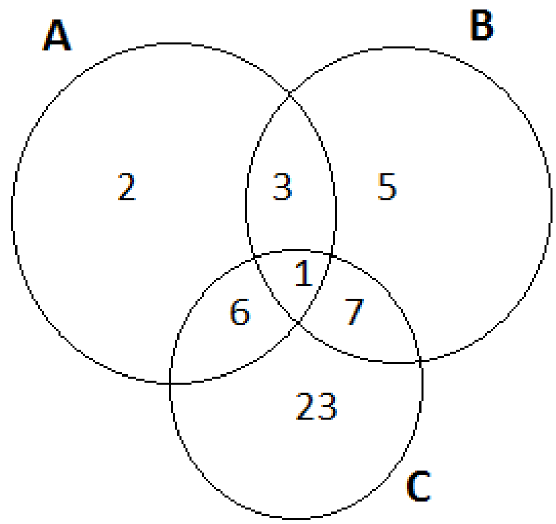


   Answer : $\frac{1}{2}$.

   The diameter of the circle is 1. The diagonal line of the inner square is 1 too. The side of the inner square is $\frac{\sqrt{2}}{2}$. Therefore, the area of the inner square is $\frac{1}{2}$.

Backup Questions

1. Find the elements of set $B' \cap C$ in the following Venn diagram (where B' denotes the complement of set B).



   Answer : $\{6, 23\}$

2. If $x = -1$ is a triple root of the polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + 2$, what is the value of a ?

   Answer : -5. Let $x_1,\ x_2,\ x_3,\ x_4$ denote the roots of polynomial $f(x)$. Since -1 is a triple root, we have $x_1 = x_2 = x_3 = -1$. Note that the product of all roots equals to 2. The forth root $x_4 = -2$. Therefor $a = x_1 + x_2 + x_3 + x_4 = -2 - 1 - 1 - 1 = -5$.

3. Simplify $\dfrac{\sqrt{5}+2}{\sqrt{5}-2} - 4\sqrt{5}$.

   Answer : 9.

   It is easy to see $\dfrac{\sqrt{5}+2}{\sqrt{5}-2} - 4\sqrt{5} = \dfrac{(\sqrt{5}+2)^2}{(\sqrt{5}+2)(\sqrt{5}-2)} - 4\sqrt{5} = (\sqrt{5}+2)^2 - 4\sqrt{5} = 9.$